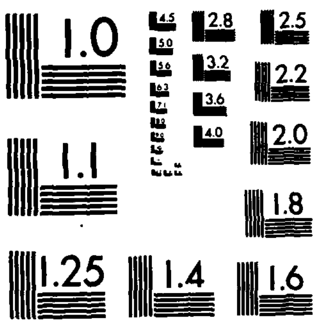
AD-A143 297 STATIC METHODS IN THE DESIGN OF NONLINEAR AUTOMATIC CONTROL SYSTEMS(U) FOREIGN TECHNOLOGY DIV WRIGHT-PATTERSON AFB OH N I ANDREYEV ET AL. 27 JUN 84 1/9

UNCLASSIFIED FTD-ID(RS)T-1734-83 F/G 12/1 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

2

FTD-ID(RS)T-1734-83



# FOREIGN TECHNOLOGY DIVISION

AIR FORCE SYSTEMS COMMAND

STATIC METHODS IN THE DESIGN OF NONLINEAR AUTOMATIC CONTROL SYSTEMS

by

N.I. Andreyev, B.G. Dostupov, et al

FOREIGN TECHNOLOGY DIVISION





Approved for public release;
distribution unlimited.

# UNEDITED MACHINE TRANSLATION

FTD-ID(RS)T-1734-83 27 June 1984

MICROFICHE NR: FTD-84-C-000637

STATIC METHODS IN THE DESIGN OF NONLINEAR AUTOMATIC CONTROL SYSTEMS

By: N.I. Andreyev, B.G. Dostupov, et al

English pages: 765

Source: Statisticheskiye Metody v Proyektirovanii Nelineynykh Sistem Avtomaticheskogo Upravleniya, Publishing House "Mashinostroyeniye", Moscow, 1970, pp. 1-407

Country of origin: USSR
This document is a machine translation.
Requester: DRSMI/YDL
Approved for public release; distribution unlimited.

| THIS TRANSLATION IS A RENDITION OF THE ORIGINAL FOREIGN TEXT WITHOUT ANY ANALYTICAL OR EDITORIAL COMMENT. STATEMENTS OR THEORIES ADVOCATED OR IMPLIED ARE THOSE OF THE SOURCE AND DO NOT NECESSARILY REFLECT THE POSITION OR OPINION OF THE FOREIGN TECHNOLOGY DIVISION. | PREPARED BY:<br><br>TRANSLATION DIVISION<br>FOREIGN TECHNOLOGY DIVISION<br>WP-AFB, OHIO. |
|---|---|

Table of Contents

## U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

| Block | Italic | Transliteration | Block | Italic | Transliteration |
|---|---|---|---|---|---|
| А а | *А а* | A, a | Р р | *Р р* | R, r |
| Б б | *Б б* | B, b | С с | *С с* | S, s |
| В в | *В в* | V, v | Т т | *Т т* | T, t |
| Г г | *Г г* | G, g | У у | *У у* | U, u |
| Д д | *Д д* | D, d | Ф ф | *Ф ф* | F, f |
| Е е | *Е е* | Ye, ye; E, e* | Х х | *Х х* | Kh, kh |
| Ж ж | *Ж ж* | Zh, zh | Ц ц | *Ц ц* | Ts, ts |
| З з | *З з* | Z, z | Ч ч | *Ч ч* | Ch, ch |
| И и | *И и* | I, i | Ш ш | *Ш ш* | Sh, sh |
| Й й | *Й й* | Y, y | Щ щ | *Щ щ* | Shch, shch |
| К к | *К к* | K, k | Ъ ъ | *Ъ ъ* | " |
| Л л | *Л л* | L, l | Ы ы | *Ы ы* | Y, y |
| М м | *М м* | M, m | Ь ь | *Ь ь* | ' |
| Н н | *Н н* | N, n | Э э | *Э э* | E, e |
| О о | *О о* | O, o | Ю ю | *Ю ю* | Yu, yu |
| П п | *П п* | P, p | Я я | *Я я* | Ya, ya |

*<u>ye</u> initially, after vowels, and after ъ, ь; <u>e</u> elsewhere.
When written as ё in Russian, transliterate as yё or ё.

## RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

| Russian | English | Russian | English | Russian | English |
|---|---|---|---|---|---|
| sin | sin | sh | sinh | arc sh | $\sinh^{-1}$ |
| cos | cos | ch | cosh | arc ch | $\cosh^{-1}$ |
| tg | tan | th | tanh | arc th | $\tanh^{-1}$ |
| ctg | cot | cth | coth | arc cth | $\coth^{-1}$ |
| sec | sec | sch | sech | arc sch | $\text{sech}^{-1}$ |
| cosec | csc | csch | csch | arc csch | $\text{csch}^{-1}$ |

| Russian | English |
|---|---|
| rot | curl |
| lg | log |

## GRAPHICS DISCLAIMER

All figures, graphics, tables, equations, etc. merged into this translation were extracted from the best quality copy available.

Page 1.

STATIC METHODS IN THE DESIGN OF NONLINEAR AUTOMATIC CONTROL SYSTEMS.

Edited by doctor of technical sciences, professor B. G. Dostupov.

Page 4.

The authors of the book:

N. I. Andreyev, doctor of technical sciences, Prof.;

B. G. Dostupov, doctor of technical sciences, Prof.;

I. Ye. Kazakov, doctor of tech. sciences, Prof.;

Yu. M. Kozlov, Cand. of tech. sciences, docent;

V. T. Kochetkov, doctor of technical sciences;

A. V. Potseluyev, Cand. of tech. sciences, docent;

Yu. V. Sayutin, Cand. of tech. sciences;

V. I. Smirnov, eng.;

N. M. Sotskiy, doctor of technical sciences;

V. I. Tikhonov, doctor of technical sciences, Prof.;

V. I. Chernektskiy, doctor of technical sciences.

Page 5.

Preface.

"The unity of nature is discovered in the "remarkable analogy" of differential equations, which relate to the different regions of phenomena".

V. I. Lenin.

A rise in productivity of labor/work, which, regarding V. I. Lenin, is "... in the latter/last calculation, most important, important itself for the conquest of the new social system", [1] is based first of all on the overall automation of production processes.

FOOTNOTE [1]. V. I. Lenin. Complete collections of works Vol. 39, page 21. ENDFOOTNOTE.

The automation of production stipulated the rapid development of the theory of automatic control, which on the eyes of one generation had time to occupy its honorary place in one series/row with the long ago prevailing "classical" sciences, after soaking into itself their best achievements and, first of all, the results, obtained in the

region the theory of differential equations. However, already it was established/installed at the glow of the development of the theory of automatic control that, besides the simplest linear systems, the nonlinear automat. control systems possess the large prospects for practical use/application. Moreover, in a number of cases they can ensure the considerably better quality of control. Therefore nonlinear automatic control systems were the object/subject of numerous experiments.

Page 6.

In the theory of nonlinear systems the problems, connected with analysis and synthesis of systems taking into account the interaction of random disturbances, at present occupies the central place. This region of research was called the statistical dynamics of nonlinear automatic control systems. Statistical analysis and synthesis of dynamic systems and, in particular automatic control systems, has great practical value because the real functioning of these systems always occurs under the conditions of the interaction of different interferences, which can have a character of external random disturbances or be random changes in the parameters of system.

Independent of the character of interferences, with the synthesis of dynamic system computed values of its parameters must be selected so

that it could in the best way fulfill its functions. For the automatic control system this requirement can be reduced, for example, to the guarantee of a minimum error in the value of the output parameter.

Beginning from the basic works of A. Ya. Khinchin in the region of the theory of random processes and work of A. N. Kolmogorov, dedicated to the solution of the problem of the synthesis of dynamic systems under the random influences and made even in the prewar period, the statistical dynamics of automatic control systems receives further development in the numerous research of Soviet and foreign scientists. Special role in the development of the statistical dynamics of automatic control systems in the postwar period belongs to V. S. Pugachev and V. V. Solodovnikov, which gave not only further development of theory, but also laid paths to its practical applications/appendices.

The tasks of the statistical theory of the transmission of signals tightly border on the classical tasks of statistical analysis and synthesis of automatic control systems. The gradual association of the methods of the solution of these problems at present is observed.

Despite the fact that the series/row of the methods of analysis

and synthesis is worked out in the region of the statistical dynamics of nonlinear systems, until this time there are no works, which reflect the variety of these methods.

The proposed to reader book, entering the six-languid series "nonlinear automatic control systems", sets as its goal to a certain extent to complete this gap/spacing. The book is the collective work, whose authors took direct part in creation and development of the new methods of analysis and synthesis of nonlinear automatic systems.

Methods of statistical analysis and synthesis of nonlinear automatic control systems and questions of the use/application of digital computers for the solution of practical problems are presented in the monograph.

Ye. Popov.

B. Dostupov.

Page 9.

## Chapter I.

## Fundamentals of statistical methods of the study of nonlinear systems.

### 1. Role of statistical methods in research of automatic control systems.

Under the actual conditions to the work of automatic control systems besides useful input signals a greater or smaller effect have all possible random disturbances (interference). Therefore the values of the output parameters of the systems, which subsequently we will call output coordinates, always differ from computed values, found for some idealized conditions for the work of system. In other words, the real dynamics of automatic control system due to the effect of random disturbances differs from ideal (calculated) dynamics.

Random disturbances with respect to the system being investigated can be external or internal.

To the external random disturbances can be attributed such,

which distort the useful input signals (input coordinates). Sometimes these disturbances/perturbations (interference) can be such considerable that the straight/direct use of an input signal together with the interference proves to be impossible.

Page 10.

In these cases it is necessary to resort to the preliminary filtration of input signal in order to decrease the interference effects.

To the functioning of automatic control system a definite effect can have also the probable deviations of the parameters, which characterize the conditions for the work of system, from their computed values (deviation of temperature, humidity, supply voltages, etc.). We will consider also such divergences external disturbances/perturbations.

To the internal random disturbances we will carry such, whose sources are laid in system itself (random noises in the radio parts, the divergence of the design parameters of system from calculated values, etc.).

For the explanation of the introduced concepts let us turn to an

example of the motion of aircraft with the autopilot. It is obvious that with the work of autopilot on each of its inputs will enter to one of the angular coordinates of aircraft, measured with certain error, which in this case plays the role of input interference. Furthermore, on the motion of aircraft they will prove to be effect also environmental factors, connected with the state of the atmosphere (wind, the fluctuation of air density, etc.). Such factors are the sources of external disturbances/perturbations. At the same time in the autopilot itself there can be the noises in the amplifiers, the divergences of gear ratios from the calculated ones, etc. These are internal random disturbances.

It is natural that research of automatic control systems in the conditions of the interaction of random noise can be realized only by probabilistic or statistical methods.

For brevity subsequently we will conditionally call all these methods statistical.

The need for the account of the effect of random disturbances on the work of automatic control systems stipulated the considerable volume of scientific theoretical and experimental studies in this region.

Especially numerous of the development of statistical methods for the linear automatic control systems. The solution of many problems in this region is led to the final results, suitable for the use in the engineering practice.

The statistical methods of analysis and synthesis of nonlinear systems are worked out to a lesser degree, although the size of the literature and in this region at present is sufficiently considerable.

The for the first time simplest tasks of the statistical calculation of nonlinear systems were examined in the works, dedicated to research of the transmission of signals in the presence of the interferences, where was investigated the transformation of random functions by inertia-free nonlinear filters in the open circuits without feedback [14].

Page 11.

For the study of the special features/peculiarities of the passage of the amplitude-modulated harmonic signal, which enters the input of the inertia-free nonlinear detector together with random noise, worked out Rice's method [79], who, however, leads to the complicated results, expressed as special functions.

It is necessary to make more precise the determination of nonlinear system from the point of view of setting statistic studies.

Nonlinear we will call such system, in which between the output coordinates and the input random disturbances there are nonlinear dependences.

During this determination the system, linear with respect to the useful input signal and some parameters, the simplest first-order system, described by the differential equation of the form

$$[(T_0+V)s+1]Y=X$$

at the initial condition Y(0)=0,

where $T_0$ - computed value of time constant;

V - probable deviation of time constant from the calculated (ideal).

This system, obviously, is linear relative to the input signal X and it is at the same time nonlinear relative to the random parameter V, that it is possible to note, after recording the solution of the

equation

$$Y \quad \int_0^t e^{-(T_0+V)(t-\tau)} X d\tau.$$

in question.

The dynamics of automatic control system can be described by the set of the differential equations, led to Cauchy's form:

$$\left.\begin{aligned}\frac{dY_1}{dt} &= f_1(Y_1, Y_2, \ldots, Y_n, X_1, X_2 \ldots, X_m, t);\\ \frac{dY_2}{dt} &= f_2(Y_1, Y_2, \ldots, Y_n, X_1, X_2, \ldots, X_m, t);\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \frac{dY_n}{dt} &= f_n(Y_1, Y_2, \ldots, Y_n, X_1, X_2, \ldots, X_m, t),\end{aligned}\right\} \quad (I.1)$$

where $Y_1, Y_2, \ldots, Y_n$ — output coordinates of system;

$X_1, X_2, \ldots, X_m$ — input coordinates.

The part of the input coordinates can be random disturbances.

Function $f_1, f_2, \ldots, f_n$ they can be both the linear and nonlinear.

Page 12.

In the more general case of equation (I.1) they can be partially

replaced by some final functional dependences between values $Y_i$ and $X_j$ or by finite-difference equations. However, whatever there was the character of these equations, in any automatic control system there is certain connection/communication between the input disturbances/perturbations and the output coordinates:

$$Y_i = A_i(t, \tau, X_1, X_2, \ldots, X_m) \quad (i = 1, 2, \ldots, n), \qquad (1.2)$$

where $A_i$ — certain functional.

We will assume that subsequently relative to disturbances/perturbations this functional is nonlinear.

Analysis and synthesis of the nonlinear automatically controlled systems, which undergo the interaction of random disturbances, is one of the most complicated regions of the contemporary theory of automatic control. At the same time this region of research in the practical sense is very important, since the nonlinear automatic systems, which possess a series/row of advantages, find ever increasing use during the control of different objects and of processes.

Hence completely understandably steady expansion and the deepening of the scientific development of questions of the theory of

nonlinear automatic control systems taking into account the effect of random disturbances. This region of the research by the statistical dynamics of nonlinear systems briefly is called.

The methods of the solution of the problems of the statistical dynamics of nonlinear systems depend on the complexity of the system, presence in it of inertial elements/cells, and also feedback. Nonlinear systems can be divided into four fundamental classes in accordance with the classification, represented in Fig. I.1.

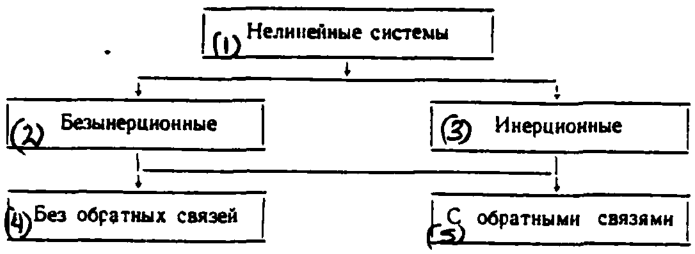



Fig. I.1. Classification of nonlinear systems.

Key: (1). Nonlinear systems. (2). Inertia-free. (3). Inertial. (4). Without feedback. (5). With by feedback.

Page 13.

The field of the tasks of the statistical dynamics of automatic control systems at present considerably was widened and it begins to encompass not only tasks of the accuracy analysis of the automatic control systems, but also of task of research of their reliability, economic efficiency, or questions of the synthesis of the optimum control systems on the base of the use of statistical criteria.

2. Characteristics of coordinates and disturbances/perturbations.

The simplest method of the representation of the random functions and parameters is the assignment of many of their

realizations respectively

$$X_s (s = 1, 2, \ldots, N) \overset{(1)}{\text{или}} V_s (s = 1, 2, \ldots, N).$$

Key: (1). or.

During the unlimited expansion of the sample size this representation in the statistical sense can be as to exact ones as desired. However, the need for the generation of a large number of realizations of real random functions or random variables is a deficiency/lack in the method. In practice can be met the cases, when this generation will be complex problem both in the fundamental and in the technical sense. Therefore frequently it is necessary to be limited to a small number of realizations, in consequence of which the representation of the random functions and parameters proves to be very inadequate.

The comprehensive characteristics of the coordinates of the automatic control system, and also input disturbances/perturbations are their distribution laws. However, obtaining such laws is conjugated/combined with the great fundamental and computational difficulties. These difficulties, when input disturbances/perturbations are not the random parameters, but the random functions of the time, any other coordinates of system or to their set, are especially great. As the obvious case to this random

function, which depends on the time and other three coordinates of system, can serve wind velocity. Any of the components of wind velocity is changed randomly on the time, and also in the dependence on the coordinates of the point of the earth's surface.

The account of the interaction of similar random disturbances is complicated by the fact that the peculiar feedback with the random transmission factor is established/installed through the disturbance/perturbation in the system.

For the explanation let us consider the axial motion of aircraft taking into account the interaction of the longitudinal random wind.

Let the aircraft move relative to the Earth with a velocity of dY/dt, and wind velocity is equal to U(t,Y). Then the velocity of aircraft v relative to air flow (airspeed) will be

$$v = \frac{dY}{dt} - U.$$

Page 14.

In the first approximation, the equation of longitudinal dynamics of aircraft can be recorded in the form

$$\frac{d^2Y}{dt^2} = T - kv^2,$$

where T - given thrust of engine;

kv² - given drag.

For simplicity of value T and k we will assume/set by constants.

Substituting for airspeed v its expression, we will obtain

$$\frac{d^2Y}{dt^2} = T - k\left[\frac{dY}{dt} - U(t, Y)\right]^2.$$

Fig. 1.2 gives the block diagram of the simulation of this equation. Components/links 1 and 2 - integrating. The feedback loop, formed by components/links 3, 4, 5, 6, includes nonlinear element 5, and also component/link 3, which possesses random transmission factor. It is obvious that the presence of this component/link substantially complicates the solution of the problems of analysis and synthesis of the system in question.

Random function X(t) can be preset with the help of the multidimensional distribution law:

$$p_x = p(x(t_1), x(t_2), \ldots, x(t_n)). \quad (1.3)$$

Since a number of points $t_i$ virtually final, this representation proves to be in the general case approximate.

The use of moments/torques of the connection/communication of the form

$$\mu_{x_K} = M[X(t_1)X(t_2)\ldots X(t_K)], \qquad (I.4)$$

where point $t_1, t_2, \ldots, t_K$ they belong to the interval of the assignment to function, is the propagated method of the assignment to random function. Especially frequently is applied the simple assignment to random function X(t) with the help of the first two moments/torques:

$$\mu_{x_1} = M[X(t_1)];$$
$$\mu_{x_2} = M[X(t_1)X(t_2)].$$

However, these characteristics only very approximately and ambiguously determine random function X(t). There is a countless set of functions X(t), possessing identical moments/torques of the first and second order.

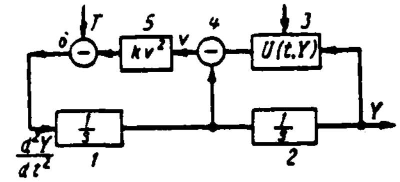



Fig. I.2. The block diagram of the simulation of the axial motion of the aircraft: 1, 2 - integrating components/links; 3 - the nonlinear element, which forms the nonlinear random function U(t,Y); 4, 6 - subtracting components/links; 5 - squaring component/link.

Page 15.

Therefore this limited determination of random disturbances is hardly suitable for the solution of the problems of the statistical dynamics of nonlinear systems. To more simply deal concerning the random parameters, time-independent and coordinates of system.

In practice frequently they attempt to avoid the straight/direct account of random functions, replacing by their all possible expansions in terms of the nonrandom functions. In these expansions the random turn out to be only the coefficients, which do not depend on the arguments of random function. It is natural that this representation, generally speaking, will be also approximate, since in the actual conditions it is necessary to be limited to a finite number of terms of the expansion of random function in terms of the

nonrandom ones.

So-called canonical expansion [77] is the propagated version of the expansions of random function in terms of the nonrandom ones, when random function X(t) is represented in the form

$$X(t)=\psi_0(t)+\sum_{j=1}^{\infty}\psi_j(t)V_j, \qquad (1.5)$$

where $\psi_j(t)\,(j=0,1,\ldots,\infty)$ — nonrandom (the so-called coordinate) functions of canonical expansion;

$V_j$ — random pair-wise uncorrelated coefficients.

It is possible to apply also all possible series/rows (Fourier, Hermite, etc.). If we bound the expansion of function by certain X(t) n term, then it will be represented by set N of random variables (parameters)

$$V_1, V_2, \ldots, V_N$$

and (N+1) nonrandom functions $\psi_0, \psi_1, \ldots, \psi_N$.

In principle many random variables $V_1, V_2, \ldots, V_N$ can be preset by its multidimensional differential law of distribution

$$p_V = p(v_1, v_2, \ldots, v_N). \qquad (1.6)$$

Instead of the differential law of distribution (I.6) for the assignment of many random variables $V_j$ it is possible to use other functions, for example, integral law $F(v_1, v_2, \ldots, v_N)$ or the characteristic function of the distribution .

$$\chi(s_1, s_2, \ldots, s_N) = \mathrm{M}\left[e^{-i(s_1V_1+s_2V_2+\ldots+s_NV_N)}\right]. \qquad (I.7)$$

However, to more simply assign the set of the moments/torques of connection/communication for values $V_j$ to certain q order:.

$$\mu_{r_1r_2\ldots r_N} = \mathrm{M}\left[V_1^{r_1}V_2^{r_2}\ldots V_N^{r_N}\right] \qquad (I.8)$$
$$(r_1+r_2+\ldots+r_N \leqslant q).$$

Page 16.

It is natural that as a result of the limitation of order q many random variables $V_j$ will be determined (preset) only approximately.

Analogous methods can be applied, also, for describing the output coordinates of system.

3. Versions of the tasks of the statistical dynamics of nonlinear systems.

The diverse variants of the formulation of the problems of the analysis of the statistical dynamics of nonlinear automatic control systems are possible depending on the method of the assignment of input random disturbances and form of the representation of the output coordinates of system. We will be bounded only to the most widely used and important in the practical sense versions.

Input disturbances/perturbations are preset in the form of many realizations $X_{js}\begin{pmatrix} j=1, 2, \ldots, m \\ s=1, 2, \ldots, N \end{pmatrix}$ of input random functions or realizations

$V_{js}\begin{pmatrix} j=1, 2, \ldots, m \\ s=1, 2, \ldots, N \end{pmatrix}$ input random parameters. This information during the infinite expansion of these sets is sufficient for determining the laws of distribution of output coordinates $Y_i(i=1, 2, \ldots, n)$ of system, and also single characteristics of these laws (moments/torques of connection/communication, etc.). Only finite number N realizations virtually can be preset, the laws of distribution of output coordinates they are determined approximately in connection with which.

Are preset the laws of distribution of input random functions

$P_x$ and parameters $P_v$. This information, as in the preceding case, it is sufficient for determining the laws of distribution of the output coordinates of system, and also their characteristics.

Are preset the moments/torques of connection/communication $M_X$ of input random functions or the moments/torques of connection/communication $M_V$ of the input random parameters. It is obvious that in the general case only at the moments/torques of connection/communication $M_X$ and $M_V$ it is not possible to restore/reduce the laws of distribution of output coordinates, and it is possible to determine (and that not always) only the single characteristics of these laws.

For the explanation let us consider the simplest example. Let the element of system be the square law detector, described by the equation

$$y = kx^2,$$

where k - constant nonrandom coefficient.

Then for determining M [Y] it suffices to assign the moment/torque

$$\mu_{2X} = M[X^2].$$

Page 17.

It is obvious that

$$M[Y] = k\mu_{2X}.$$

However, for determining the dispersion D [Y] it is necessary, besides $\mu_{2X}$, to assign even and the moment/torque

$$\mu_{4X} = M[X^4],$$

since

$$D[Y] = M[Y^2] - (M[Y])^2 = M[k^2X^4] - (k\mu_{2X})^2 = k^2(\mu_{4X} - \mu_{2X}^2).$$

There are systems, for which with the fact or another approximation/approach the law of distribution of the output coordinate Y can be determined according to the characteristics (but not to the distribution laws) of input random disturbances. For example, in the linear systems with a large number of small input disturbances/perturbations, which operate independently and having one order of smallness, the law of distribution of output coordinate can be close to the normal despite the fact that the laws of

distribution of input random disturbances can be distant from the normal ones. Analogous effect can sometimes be observed, also, in the nonlinear systems. However, this question is not thus far yet worked out.

The classification of the basic versions of the tasks of the statistical dynamics of nonlinear systems depending on the form of the assignment of the input random disturbances X and V and the form of the representation of the output coordinates Y is given in table I.1. In this table signs "+" and "X" designated the versions of tasks, which make sense. The special versions of the tasks, when the law of distribution of output coordinate must be determined at the moments/torques of the connection/communication of input disturbances/perturbations, are designated by sign "X". In all in the table are contained 16 versions of tasks. Let us note that each version of task can be realized for any of four classes of the nonlinear systems, designated in Fig. I.1.

Table I.1.

| Форма представления выходных координат (1) | (2) Форма задания входных случайных функций и параметров | | | | | |
|---|---|---|---|---|---|---|
| | $X_U$ | $V_U$ | $P_X$ | $P_V$ | $M_X$ | $M_V$ |
| Реализации $Y$ (3) | + | + | + | + | | |
| Законы распределения $P_Y$ (4) | + | + | + | + | × | × |
| Моменты $M_Y$ (5) | + | + | + | + | + | + |

Key: (1). Form of the representation of output coordinates. (2). Form of assignment of input random functions and parameters. (3). Realizations. (4). Distribution laws. (5). Moments/torques.

Page 18.

Thus, in the full/total/complete examination of the tasks of the statistical dynamics of nonlinear systems within the framework of the classification accepted necessary to investigate 16x4=64 the diverse variants of tasks which must be further varied depending on the stated goal (analysis or the synthesis of system), the form of the nonlinear elements/cells, connected with the system, the type of feedback, order of system, etc. It is obvious that this presentation of the statistical theory of the dynamics of nonlinear automatic control systems would be inadmissible to bulky ones and tiresome

ones. Since one and the same method, as a rule, is applied for solving the series of problems of statistical dynamics, is more expedient to use a classification not of tasks themselves, but methods of their solution. In this case the presentation of theory acquires large purposefulness and generality.

Let us consider the fundamental methods of the statistic studies of nonlinear systems briefly, bearing in mind that their comprehensive illumination is given in the appropriate chapters.

4. Fundamental methods of the statistic studies of nonlinear systems.

Method for statistical testing (Monte Carlo method) [15, 16]. At present the method for statistical testing solidly entered into many regions of scientific research. It obtained use/application, also, during the solution of the problems of the statistical dynamics of nonlinear automatic control systems. Its essence is reduced to the input/introduction of the random realizations of input random functions $X_{\text{н}}$ or parameters $V_{\text{н}}$ to the appropriate inputs of the system being investigated.

One realization of input disturbance/perturbation must be given for each of the inputs of system during one testing, in this case the realization of each of the output coordinates will be obtained.

Repeating similar tests repeatedly, we will obtain the set of realizations for each of the output coordinates. Subjecting further these sets to statistical processing, we determine the laws of distribution of output coordinates or that are simpler, the single characteristics of these laws.

The physical or mathematical simulation of the random functions and parameters is applied for reproduction and input/introduction of input disturbances/perturbations together with the use of real recordings of their realizations. For this purpose is created a large number of diverse physical sensors of random functions and random variables, and also programs for obtaining on the computer(s) of the so-called pseudorandom numbers, on base of which are synthesized the realizations of the random functions (see Chapter XI).

Page 19.

Universality and simplicity are the obvious advantages of the method for statistical testing. Method can be used in connection with any nonlinear systems, the fundamental complexity of method not depending on the complexity of the very control system.

Method allows/assumes the use not only of mathematical models of systems, but also half-scale models, which contain the single units

of system. In the principle the method can be realized directly on system itself, if only it is technically possible the input/introduction of different random disturbances into system and change in its parameters.

The need for the accumulation of the large files of information about the output coordinates of system at the same time is a deficiency/lack in the method for statistical testing, which is connected with the fulfillment of the considerable space of computations. In order to obtain the laws of distribution of the output coordinates of system or at least their single characteristics with the accuracy acceptable for the practice, are required to compute hundred even thousands of values of these coordinates.

In this case one should stress that with an increase in the size of statistical sample together with an increase in the degree of confidence in the correctness of the determination of result it will always remain and the certain degree of the risk of obtaining erroneous data.

The necessary evaluations/estimates can be made on the known statistical criteria. Without stopping on this, let us only stress that it is desirable, after preserving simplicity and universality of the method for statistical testing, to decrease the space of the

necessary computations and to be freed from the statistical uncertainty/indeterminacy of the obtained results. To a certain extent the methods of the equivalent disturbances/perturbations (see Chapter IV, V, VI) satisfy these requirements.

Methods of equivalent disturbances/perturbations. The essence of these methods is reduced to the fact that instead of the random realizations of parameters $V_j$, of the utilized in the method statistical tests, previously are designed nonrandom values $\xi_{js}$ (j=1, 2, ..., m; s=1, 2, ..., N), called equivalent disturbances/perturbations.

Equivalent disturbances/perturbations $\xi_{js}$ are input/embedded to the appropriate inputs of the nonlinear system being investigated, in this case by computations or simulation are determined some values $Y_s$ of output coordinate. The unknown probabilistic characteristics of coordinate Y are formed/shaped from values $Y_s$

It is obvious that the fundamental task, which appears in the methods of equivalent disturbances/perturbations, is this determination of values $\xi_{js}$, with which it would be provided simplicity of the computation of the unknown probable characteristics of the output coordinates of system it would be required a comparatively small number of solutions of the equations of the control system

being investigated.

The aforesaid let us clarify by the simplest example.

Page 20.

Let us assume that is examined the system with one random parameter V, functional connection of which with the output coordinate can be with a sufficient accuracy described by the quadratic polynomial of the form

$$Y = A + BV + CV^2, \qquad (I.9)$$

where A, B and C - nonrandom constant or variable coefficients.

Let us pose the problem of determining the mathematical expectation M [Y].

If for the solution of this problem of using the method for statistical testing, then it is necessary to input/embed the random realizations of parameter V and after accumulating statistical material to value Y, further from the ordinary formulas of statistical processing of values to find M [Y].

Let us solve this task by the method of equivalent

disturbances/perturbations.

Let us assume for simplicity that M [V] =0. This assumption, obviously, will not break the generality of reasonings.

Passing in recorded functional dependence (I.9) to the right and to the left of the random variables to their mathematical expectations we will obtain

$$M[Y] = A + C\sigma_V^2, \qquad (I.10)$$

where $\sigma_V$ — the root-mean-square deviation of value V.

In order to compute M [Y] according to this formula, it suffices initial dependence (I.9) to substitute only two different values of value V, namely:

$$\xi_1 = +\sigma_V;$$
$$\xi_2 = -\sigma_V.$$

As a result of such substitutions we will obtain

$$Y_1 = A + B\sigma_V + C\sigma_V^2;$$
$$Y_2 = A - B\sigma_V + C\sigma_V^2.$$

After determining the average of values $Y_1$ and $Y_2$, let us find

$$\frac{Y_1+Y_2}{2}=A+C\sigma_V^2.$$

Comparing this expression with expression (I.10), obviously, we can record

$$M[Y]=\frac{Y_1+Y_2}{2}.$$

Page 21.

Thus, as a result only of the two substitutions (but not hundred or thousand, as in the method for statistical testing) is obtained the exact value M [Y]. Values $\xi_1$ and $\xi_2$ in this case are equivalent disturbances/perturbations.

On the basis of this example it is possible to do only the conclusion that there are tasks, for solving which the method of equivalent disturbances/perturbations proves to be more effective than the method for statistical testing.

In actuality there are borders, with which the method of equivalent disturbances/perturbations retains its advantages [34].

The method of determining the equivalent disturbances/perturbations presented is not single. Depending on the method accepted appear the numerous varieties of the methods of this group.

Method of statistical linearization. At the basis of the method of statistical linearization lies/rests the idea about the possibility of this replacement of the nonlinear elements of the automatic control system by some linear components/links, in which the statistical characteristics of output coordinates would coincide with the analogous characteristics of nonlinear elements or they would be close to them.

For the explanation let us consider the nonlinear element, which is described by the equation

$$Y = F(X), \qquad (I.11)$$

where F - nonlinear function.

Let us assume that law p(x) of distribution of the random input coordinate X is preset. Then the mathematical expectation of the output coordinate Y is equal

$$a_Y = M[Y] = \int_{-\infty}^{+\infty} F(x)\,p(x)\,dx, \qquad (I.12)$$

but the dispersion

$$D_Y = D[Y] = M[Y^2] - a_Y^2 = \int_{-\infty}^{+\infty} F^2(x)\,p(x)\,dx - a_Y^2. \qquad (I.13)$$

Let us replace nonlinear element with certain linear, after preserving input and output coordinates, i.e., let·us take the component/link, described by the equation

$$Y = kX, \qquad (I.14)$$

let us select coefficient of k so that the moments/torques of the first and second orders would preserve their values, i.e., they would be as before equal to $a_Y$ and $D_Y$.

Page 22.

On the basis of equality (I.12) and (I.14) let us find

$$a_Y = ka_X = \int_{-\infty}^{+\infty} F(x)\,p(x)\,dx;$$

hence

$$k=\frac{\int\limits_{-\infty}^{+\infty}F(x)\,p(x)\,dx}{a_X}. \qquad (I.15)$$

Further from expressions (I.13) and (I.14) we will obtain

$$D_Y=k^2D_X=\int\limits_{-\infty}^{+\infty}F^2(x)\,p(x)\,dx-a_Y^2;$$

$$k^2=\frac{\int\limits_{-\infty}^{+\infty}F^2(x)\,p(x)\,dx-a_Y^2}{D_X}. \qquad (I.16)$$

It is obvious that the values of coefficient of k, found from formulas (I.15) and (I.16), in the general case do not coincide, i.e., for computing of mathematical expectations and dispersions it is necessary to choose different linear components/links. However, this is not so essential a difficulty. That is important, coefficients k can be determined for the preset form of nonlinearity F(x) and the preset law of distribution of input coordinate, i.e., the statistical linearization of nonlinear elements is possible.

Let us note that it is possible to require instead of the register of the moments/torques of the output coordinates of nonlinear and linear components/links so that only the sufficient approximation/approach of these coordinates would be realized. One or the other variety of the methods of statistical linearization can be

obtained depending on the selected method of approximation/approach. Since in the real automatic control systems usually are linear and nonlinear elements, the methods of statistical linearization give the possibility to replace the nonlinear systems with linear ones, to which can be used the detailed methods of analysis and synthesis of linear systems. In this the fundamental advantage of the methods of statistical linearization.

The moments/torques of higher orders can be taken into consideration in principle during the statistical linearization.

However, during the practical use of methods of statistical linearization can arise the difficulties, connected with the fact that law p(x) of distribution of coordinate X is not always known previously, but it must be determined in the process of research of automatic control system. This difficulty, which is especially developed during the research of looped systems, nevertheless can be surmounted. In more detail this question will be examined in chapter II.

The methods of statistical linearization are generalized both to the components/links with many input parameters and also to the inertial components/links.

Page 23.

Exact analytical methods. Weight the enumerated above methods in the general case are approximate. Of great interest, especially for the theory, is the development of the exact methods of the statistical dynamics of nonlinear systems. Unfortunately, this task proves to be extremely complicated. Especially it is complicated in that version of its setting, when input disturbances/perturbations are preset in the form of random functions, i.e., when even the exact assignment of quite input disturbances/perturbations in general form is not worked out. Therefore we will be bounded to the simpler version of setting the task, when to the inputs of the nonlinear system, described by equations (I.1), are fed not random functions $X_j$ (j=1, 2, ..., m), but random parameters $V_j$ (j=1, 2, ..., m). Let us assume also that the law

$$p_0(y_{10}, y_{20}, \ldots, y_{n0}, v_1, v_2, \ldots, v_m)$$

is known allocation of parameters $V_j$ and initial conditions

$$Y_i(0) = Y_{i0} \ (i = 1, 2, \ldots, n).$$

Upon this setting it is possible to determine the law of distribution

$$p(y_1, y_2, \ldots, y_n, v_1, v_2, \ldots, v_m)$$

of output coordinates $y_i$ (i=1, 2, ..., n) system in the form of the integral of certain linear differential equation in the partial first-order derivatives, which has form [27]

$$\frac{\partial p}{\partial t}+\sum_{i=1}^{n} f_i \frac{\partial p}{\partial y_i}+p\sum_{i=1}^{n}\frac{\partial f_i}{\partial y_i}=0, \qquad (I.17)$$

where $f_i$ — nonlinear functions, which figure in the system of equations (I.1), which corresponds to the automatic control system being investigated.

Unfortunately, the cases, when the exact solution of equation (I.17) can be obtained, virtually are encountered extremely rarely. In general form to integrate this equation is possible only with the help of the approximate numerical methods, i.e., is created the position, in which the task, solved in principle accurately, cannot be led to the accurate numerical results, and method as the final result is approximate. Moreover, if we take into account that during the solution of equation (I.17) it is possible to use the equivalent system of ordinary differential first-order equations, which in this case coincides with system (I.1), then further procedure of computations will be reduced to substitution into system (I.1) instead of the input random parameters of the series/row of their

nonrandom realizations. This procedure of computations is analogous to the application of the method of equivalent disturbances/perturbations. In more detail the development of method examined in chapter VII.

end section.

Page 24.

Chapter II.

Calculation of control processes in nonlinear systems by the method of statistical linearization.

1. Statistical linearization of nonlinearity.

In this chapter we will be bounded to the examination of the widespread class of the control systems with the lumped parameters, which work in the continuous or discrete modes/conditions during the random disturbances. In the theoretical examination all nonlinear properties of such systems are conveniently related to the inertia-free nonlinear elements/cells (components/links), characterized by the functional connections between the variables without the time lag. These functional connections between the variables of nonlinear elements/cells are called their characteristics, and elements/cells themselves - nonlinearity. Signal lag during the transformation by its system let us relate completely to the linear inertial parts (components/links). Consequently, any

continuous or discrete system can be considered consisting of the connect/joined together linear systems and the nonlinear elements/cells.

Page 25.

Nonlinearity in the automatic systems with a small range of a change in the input signal can be replaced by nonlinear elements/cells. On the high input signal level in the nonlinear elements/cells their substantially nonlinear properties (for example, the limitation of output signal) are developed. In these cases it is necessary during the calculation of system to take into account the effect of nonlinear characteristics or to design system so that the processes in the system would proceed in the range of their linearity. The part of the elements/cells has in principle nonlinear characteristics with any input signals. In this case for some nonlinearity output variable can depend not only on the value of input variable, but also on the direction of its change (ambiguous characteristic). The presence of such fundamental nonlinearity in the system changes not only the quantitative characteristics of processes in it, but it is possible to lead to the appearance of the qualitatively new processes, which are not characteristic to linear system.

Let us consider the tasks of the evaluation of the accuracy of

reproduction and transformation of the preset or random useful signals in the presence of interferences, accuracy of control processes under the effect of random disturbances and with a random change in the parameters of system, and also task of the study of oscillations during the random disturbances. Let us use the method of statistical linearization for the solution of the enumerated problems in the nonlinear systems.

In the automatic systems is most widely used single-valued inertia-free nonlinearity, characterized by the nonlinear dependence of the general view between by the variables Y and X:

$$Y = F(X). \qquad \text{(II.1)}$$

Let the input random signal X(t) be represented in the form of the sum of mathematical expectation $m_X(t)$ and central component $X^{\circ}(t)$, i.e.

$$X(t) = m_X(t) + X^{\circ}(t). \qquad \text{(II.2)}$$

The method of the statistical linearization of nonlinearity is of the replacement of the nonlinear function F the linearized dependence equivalent in the probability sense between random variables $X^{\circ}$ and Y:

$$Y \approx F_0 + k_1 X^{\circ}. \qquad \text{(II.3)}$$

Function $F_0$ is the statistical (average/mean) characteristic of nonlinearity; $k_1$ - statistical factor of amplification of nonlinear element/cell on random central component. For the odd functions F statistical characteristic can be represented in the form

$$F_0 = k_0 m_X, \qquad (II.4)$$

where $k_0$ - statistical factor of amplification of nonlinearity on the mathematical expectation of input signal.

Page 26.

Function $F_0$ and coefficients $k_0$, $k_1$ are determined from the condition of the probabilistic equivalency of functions (II.1) and (II.3). The criterion of the minimum of root-mean-square error of the approximation is simplest:

$$\alpha_2 = M\{[Y - F_0 - k_1 X^0]^2\} = \min, \qquad (II.5)$$

where M - operator of mathematical expectation.

We convert expression (II.5) to the form

$$\alpha_2 = M[Y^2] + F_0^2 + k_1^2 D_X - 2F_0 m_Y - 2k_1 M[YX^0] = \min. \qquad (II.6)$$

With the the known to $M[Y^2]$, $D_X$, $m_Y$, $M[YX^0]$ value $\alpha_2$ is the function of parameters $F_0$, $k_1$. Calculating particular derivatives of $\alpha_2$ on $F_0$ and $k_1$ and equalizing to their zero, we obtain

$$F_0 = m_Y; \qquad (II.7)$$

$$k_1 = \frac{1}{D_X} M[YX^0]. \qquad (II.8)$$

It is easy to demonstrate that dependences (II.7) and (II.8) provide the minimum of error $\alpha_2$ [34].

For computing the functions $F_0$ and $k_1$ it is necessary to know the one-dimensional probability density p(x, t) of random function X(t). In this case formulas (II.7) and (II.8) take the form

$$F_0 \int_{-\infty}^{\infty} F(x)\,p(x, t)\,dx; \tag{II.9}$$

$$k_1 = \frac{1}{D_X} \int_{-\infty}^{\infty} F(x)(x - m_X)\,p(x, t)\,dx. \tag{II.10}$$

The density function of probability p(x, t) in the general case is previously unknown. It is at the same time established/installed [34, 69], that a change in the form of the law of distribution p(x, t) does not exert a substantial influence on function $F_0$ and coefficient $k_1$. More essential proves to be their dependence on mathematical expectation $m_X$ and dispersion $D_X$ of random function X(t). Therefore in the method of statistical linearization admissibly to approximately determine function $F_0$ and statistical factors of amplification $k_0$, $k_1$ for the equivalent normal distribution law with the parameters, equal to $m_X$, $D_X$. Further foundation for this assumption is also the fact that the nonlinear elements/cells in the

complex control systems are connected with the inertial linear networks, which normalize the law of distribution of signal. Let us take the law of density distribution of probability in the form

$$p(x) = \frac{1}{\sqrt{2\pi D_x}} \exp\left[-\frac{(x-m_x)^2}{2D_x}\right]. \qquad (II.11)$$

Page 27.

The statistical characteristic $F_0$ and statistical factors of amplification $k_0$, $k_1$ for any nonlinear element/cell now can be calculated previously and expressed through parameters $m_x$ and $D_x$ of input signal.

Besides the nonlinear elements/cells of the simplest form, described by equation (II.1), in the automatic control systems multiplicative and other elements/cells with the more complicated ambiguous characteristics are applied. Finally, the parameters of nonlinear element/cell can be random. Therefore in the general case inertia-free nonlinear control components can be considered as multidimensional functional dependences with nonadditive correlated input signals $X_1, \ldots, X_n$ [32]:

$$Y = F(X_1, \ldots, X_n), \qquad (II.12)$$

where F - arbitrary single-valued nonlinear function.

By the equations of form (II.12) are described also ambiguous

nonlinearity, for example hysteresis type, the depending on the direction changes in the input signal, if we in a number of arguments include/connect derivative by this variable. For the statistical linearization of nonlinearity (II.12) let us represent output variable in the form

$$Y = F_0 + \sum_{i=1}^{n} k_i X_i^0, \quad i = 1, \dots, n. \qquad \text{(II.13)}$$

The statistical characteristic $F_0$ and statistical factors of amplification $k_i$ let us determine from the condition of the minimum of root-mean-square error of the approximation

$$M\left\{\left[F(X_1, \dots, X_n) - F_0 - \sum_{i=1}^{n} k_i X_i^0\right]^2\right\} = \min. \qquad \text{(II.14)}$$

Applying the conditions of the extremum of expression (II.14) from parameters $k_i$ and $F_0$, we will obtain equations [32]

$$F_0 = M[F(X_1, \dots, X_n)]; \qquad \text{(II.15)}$$

$$\sum_{i=1}^{n} k_i D_{ij} = D_{F_j}, \quad j = 1, \dots, n, \qquad \text{(II.16)}$$

where $D_{ij} = M[X_i^0(t)\, X_j^0(t)]$; $D_{F_j} = M[F X_j^0(t)]$.

Solving system of equations (II.16) relative to $k_i$, we will obtain

$$k_i = \sum_{j=1}^{n} (-1)^{i+j} \frac{\Delta_j^i}{\Delta} D_{F_j}, \quad i = 1, \dots, n, \qquad \text{(II.17)}$$

where $\Delta$ - determinant of system (II.16);

$\Delta_j^i$ - the cofactor of the element/cell of the $i$ column of the j line of determinant Δ.

Page 28.

In the special cases of the uncorrelated variables, entering dependence (II.12), formulas (II.17) are simplified:

$$k_i = \frac{M\left[FX_i^0\right]}{D_{ii}}, \quad i = 1, \ldots, n. \qquad \text{(II.18)}$$

Let us assume on the basis of the same considerations, which were expressed above, that the probability density of the joint distribution of variables $X_1, \ldots, X_n$ is normal:

$$p(x_1, \ldots, x_n) = \frac{1}{\sqrt{2^n \pi^n \Delta}} \exp\left[\frac{\Delta^*}{2\Delta}\right], \qquad \text{(II.19)}$$

where Δ* - bordered definition, obtained from Δ via adding of one n+1 column and n+1 line of terms $x_1 - m_{x_1}, \ldots, X_n - m_{x_n}, 0$.

Differentiating expression (II.15) for function $F_0$ on $m_{x_i}$ under the normal law of distribution (II.19) and comparing result with formula (II.17), we will obtain new expression for the coefficients of statistical linearization [32, 34, 76]:

$$k_i = \frac{\partial F_0}{\partial m_{x_i}}, \quad i = 1, \ldots, n. \qquad \text{(II.20)}$$

Let us consider examples of nonlinearity and their statistically linearized equivalents.

Example 1. Nonlinearity of the type of ideal relay (Fig. II.1) Y=h sign X. The linearized equivalent takes form $Y=F_0+k_1X^0$, where $F_0=k_0m_x$ in view of the oddness of characteristic. For computing the function $F_0$ we use formula (II.9) under the normal law of distribution (II.11) of probability density p(x):

$$F_0=-\frac{h}{\sqrt{2\pi D_X}}\int_{-\infty}^{0}\exp\left\{-\frac{(x-m_X)^2}{2D_X}\right\}dx+$$
$$+\frac{h}{\sqrt{2\pi D_X}}\int_{0}^{\infty}\exp\left\{-\frac{(x-m_X)^2}{2D_X}\right\}dx.$$

After the replacement of variable $x-m_X=t\sqrt{D_X}$ we will obtain

$$F_0=\frac{h}{\sqrt{2\pi}}\left\{\int_{-\frac{m_X}{\sqrt{D_X}}}^{\infty}e^{-\frac{t^2}{2}}dt-\int_{-\infty}^{-\frac{m_X}{\sqrt{D_X}}}e^{-\frac{t^2}{2}}dt\right\}=2h\Phi\left(\frac{m_X}{\sqrt{D_X}}\right). \quad \text{(II.21)}$$

where $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{t^2}{2}}dt$ - function of Kramp, the table of values of which is given in the application/appendix.

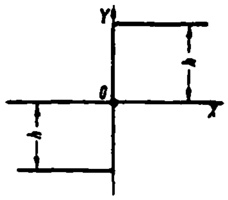



Fig. II.1. Relay ideal characteristic.

Page 29.

Coefficient $k_0$ is calculated from formula $k_0 = \frac{F_0}{m_X}$. For determining the coefficient let us use formulas (II.10) at the normal probability density p(x) or formula (II.20). As a result we obtain

$$k_1 = \frac{2h}{\sqrt{2\pi D_X}} \exp\left[-\frac{m_X^2}{2D_X}\right]. \qquad (II.22)$$

Example 2. Nonlinearity of the type of quadratic function (Fig. II.2) $Y=X^2$. The linearized equivalent takes form $Y=F_0+k_1X^0$. Function $F_0$ in this case on the basis of formulas (II.7) or (II.9) takes the form

$$F_0 = m_X^2 + D_X. \qquad (II.23)$$

For the determination of coefficient $k_1$ let us use formula (II.20). As a result we obtain

$$k_1 = 2m_X. \qquad (II.24)$$

Example 3. Ambiguous nonlinearity (Fig. II.3). This dependence

can be recorded in the single-valued form, if we into number of arguments introduce the derivative $\dot{X}$:

$$Y = F(X, \dot{X}) = \begin{cases} h & X > C, \quad \dot{X} \gtrless 0, \\ & -C < X < C, \quad \dot{X} < 0; \\ -h & -C < X < C, \quad \dot{X} > 0, \\ & X < -C, \quad \dot{X} \lessgtr 0. \end{cases} \qquad (11.25)$$

Let us designate the variable $X = X_1$ and $\dot{X} = X_2$. Let us record the linearized dependence in the form

$$Y \approx F_0 + k_1 X_1^0 + k_2 X_2^0.$$

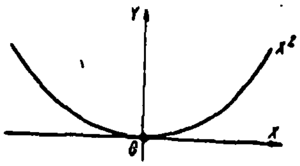



Fig. II.2.

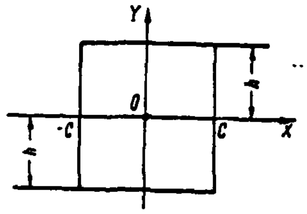



Fig. II.3.

Fig. II.2. Characteristic of square-law function generator.

Fig. II.3. Ambiguous relay characteristic.

Page 30.

For computing the function $F_0$ let us use formula (II.15):

$$F_0 = h\int_C^{\infty} p(x_1)\,dx_1 - h\int_{-\infty}^{-C} p(x_1)\,dx_1 + h\int_{-C}^{C} dx_1 \times \left[\int_{-\infty}^{0} p(x_1, x_2)\,dx_2 - \int_0^{\infty} p(x_1, x_2)\,dx_2\right], \tag{II.26}$$

where

$$p(x_1, x_2) = \frac{1}{2\pi\sqrt{D_{X_1}D_{X_2} - D_{X_1X_2}^2}} \times e^{-\frac{(x_1-m_{X_1})^2 D_{X_2} + (x_2-m_{X_2})^2 D_{X_1} - 2D_{X_1X_2}(x_1-m_{X_1})(x_2-m_{X_2})}{2(D_{X_1}D_{X_2} - D_{X_1X_2}^2)}},$$

$$p(x_1) = \int_{-\infty}^{\infty} p(x_1, x_2)\,dx_2.$$

After the fulfillment of the operations of integration we obtain odd function $m_x$:

$$F_0 = h\left\{\Phi\left(\frac{C-m_{X_1}}{\sqrt{D_{X_1}}}\right) - \Phi\left(\frac{C-m_{X_1}}{\sqrt{D_{X_1}}}\right) - 2\Phi\left(\frac{m_{X_2}}{\sqrt{D_{X_2}}}\right) \times \right.$$
$$\times\left[\Phi\left(\frac{C-m_{X_1}}{\sqrt{D_{X_1}}}\right) + \Phi\left(\frac{C+m_{X_1}}{\sqrt{D_{X_1}}}\right)\right] +$$
$$\left. + \frac{D_{X_1X_2}}{\pi\sqrt{D_{X_1}D_{X_2}}}\left[e^{-\frac{(C-m_{X_1})^2}{2D_{X_1}}} - e^{-\frac{(C+m_{X_1})^2}{2D_{X_1}}}\right]\right\}. \quad \text{(II.27)}$$

Statistical factors of amplification $k_0$, $k_1$ and $k_2$ are determined from the formulas

$$k_0 = \frac{F_0}{m_X}; \; k_1 = \frac{\partial F_0}{\partial m_{X_1}}; \; k_2 = \frac{\partial F_0}{\partial m_{X_2}}.$$

Example 4. Multiplicative type nonlinearity $Y=X_1X_2$. Statistically the linearized equivalent takes form $Y=F_0+k_1X^0{}_1+k_2X^0{}_2$. For function $F_0$ and coefficients $k_1$ and $k_2$ in this case we obtain the formulas

$$F_0 = m_{X_1}m_{X_2} + D_{X_1X_2}; \; k_1 = m_{X_2}; \; k_2 = m_{X_1}. \quad \text{(II.28)}$$

Example 5. Nonlinearity of the type of ideal relay with the random level of the output signal $Y=X_1$ sign $X_2$. Statistically linearized equivalent

$$Y \approx F_0 + k_1X_1^0 + k_2X_2^0.$$

Formula for determining the function $F_0$ in this case takes the

form

$$F_0 = -\int_{-\infty}^{0} dx_2 \int_{-\infty}^{\infty} x_1 p(x_1, x_2)\, dx_1 + \int_{0}^{\infty} dx_2 \int_{-\infty}^{\infty} x_1 p(x_1, x_2)\, dx_1. \qquad (II.29)$$

After the fulfillment of the necessary transformations we obtain

$$F_0 = \frac{2m_{x_1}}{\sqrt{QD_{x_1}}} \Phi\left(m_{x_1}\sqrt{Q}\right) + \frac{2D_{x_1x_2}}{D_{x_1}\sqrt{2\pi D_{x_1} Q}} e^{-\frac{Qm_{x_1}^2}{2}}, \qquad (II.30)$$

where

$$Q = \frac{D_{x_1}}{\Delta} + \frac{D_{x_1x_2}}{D_{x_1}\Delta};\ \Delta = D_{x_1}D_{x_2} - D_{x_1x_2}.$$

Coefficients $k_1$ and $k_2$ are determined by the differentiation of function $F_0$ on $m_{x_1}$ and $m_{x_2}$ respectively.

Page 31.

2. Calculation of the accuracy of the nonlinear stationary continuous control system.

The mode/conditions, which was steady after transient decay under the continuous influence of stationary random disturbances, is one of the characteristic modes/conditions of the functioning of the fixed systems of control. In many cases during the design of system this mode/conditions can be considered calculated. Therefore has sense to consider it in more detail. In the general case the control system can contain several inputs and outputs. For the simplification let us consider system with one output Y(t) and one input U(t). At

the input of system operate the determined useful signal f(t) and the additive stationary random noise n(t) with the zero mathematical expectation:

$$U(t)=f(t)+n(t). \qquad (II.31)$$

Nonlinear systems frequently are designed as a whole as linear. In other cases of the system of control they are projected/designed as nonlinear, in which are included the nonlinear inertia-free elements/cells, which do not have linear zones. Such systems include the stabilization systems and followers with the nonlinea. actuating elements/cells and nonlinear equalizers. Dynamic processes in these systems can be more complicatedly than in the linear systems. The onset of the auto-oscillations of the specific amplitude and frequency is possible in the steady-state mode/conditions. We will assume that they are absent in the steady-state mode/conditions of auto-oscillation. The analysis of auto-oscillations during the random disturbances let us lead below. The task of each systems consists of the reproduction on the output of useful signal or certain function from it, in particular, of the maintenance of the constant value of output variable.

We will consider that the nonlinear system is intended for obtaining on the output of signal $Y_T(t)=Lf(t)$, where L - certain linear operation:

$$Lf(t) \quad \alpha_0 f(t) + \alpha_1 \frac{df}{dt} + \ldots + \alpha_n \frac{d^n f}{dt^n}. \qquad (II.32)$$

In particular, for the servo system $\alpha_0 1$; $\alpha_1=\alpha_2=\ldots=\alpha_n=0$. The guidance dispersion is determined by the expression

$$\varepsilon(t)=Lf(t)-Y(t),$$

where Y(t) - actual output signal.

Page 32.

Let us assume that for certainty the system contains one-dimensional nonlinear element/cell with the odd characteristic and has equations

$$\left.\begin{array}{c} E_1(s)Y=H_1(s)F(X); \\ E_2(s)X=H_2(s)(U-Y_1); \\ E_3(s)Y_1=H_3(s)Y; \end{array}\right\} \quad (II.33)$$

where $E_1(s)$; $E_2(s)$; $E_3(s)$; $H_1(s)$; $H_2(s)$; $H_3(s)$ - polynomials relative to s with the constant coefficients; F(X) - nonlinear function. The structural scheme of system is depicted in Fig. II.4. After the replacement of function $F(X)$ statistical linear equivalent $F(X)=k_0(m_X, D_X)m_X+k_1(m_X, D_X)X^0$ of equation (II.33) let us record in the form of two connected systems

$$\left.\begin{array}{c} E_1(s)m_Y=H_1(s)k_0m_X;\ E_2(s)m_X=H_2(s)(f-m_{Y_1}); \\ E_3(s)m_{Y_1}=H_3(s)m_Y;\ k_0=k_0(m_X, D_X); \end{array}\right\} \quad (II.34)$$

$$\left.\begin{array}{c} E_1(s)Y^0=H_1(s)k_1X^0;\ E_2(s)X^0=-H_2(s)Y_1^0; \\ E_3(s)Y_1^0=H_3(s)Y^0;\ k_1=k_1(m_X, D_X). \end{array}\right\} \quad (II.35)$$

Equations are formally linear with the the known $k_0$ and $k_1$ and determine respectively mathematical expectations and random components of signals in the system. However, they are connected through values $m_x, D_x$, entering coefficients of $k_0$ and $k_1$. In the steady-state mode/conditions, if such in the system exists, the values of mathematical expectation $m_x$ and dispersion $D_x$ are constant. Consequently, coefficients $k_0$ and $k_1$ are constant, and systems as a whole are stationary. The theory of linear fixed systems for determining of mathematical expectations and dispersions of variables in steady-state mode/conditions [34], [76] is applicable in this case to them.

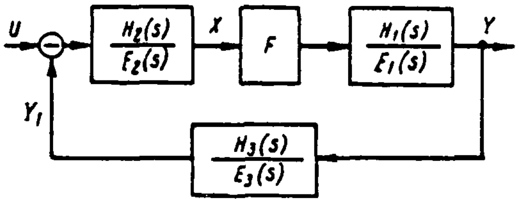


Fig. II.4. Structural scheme of nonlinear system.

Page 33.

Following this theory, for the steady-state mode/conditions from the system of equations (II.34) we will obtain expressions for the mathematical expectations of output signal and error:

$$m_Y(t) = \sum_{r=1}^{\infty} \frac{1}{r!} W_{0Y}^{(r)}(0) f^{(r)}(t) + W_{0Y}(0)\, m_n; \qquad \text{(II.36)}$$

$$m_\varepsilon(t) = \sum_{r=1}^{\infty} c_r f^{(r)}(t) - W_{0Y}(0)\, m_n, \qquad \text{(II.37)}$$

where $c_r = \alpha_r - \frac{1}{r!} W_{0Y}^{(r)}(0), \quad r = 0, 1, \ldots$ - error coefficients; $f^{(r)}(t)$ - derivatives of the r order of useful signal; $m_n$ - constant value (mathematical expectation of interference); $W_{0Y}^{(r)}(0)$ - derivatives of the transfer function of system (II.34), calculated from input U(t) to output Y(t) with argument s=0. Transfer function $W_{0Y}(s)$ is obtained for the steady-state mode/conditions from equations (II.34):

$$W_{0Y}(s) = \frac{k_0 H_1(s) H_2(s) E_3(s)}{E_1(s) E_2(s) E_3(s) + k_0 H_1(s) H_2(s) H_3(s)}. \qquad \text{(II.38)}$$

For computing the coefficients of $k_0$ and $k_1$ will be required also the formula for $m_X$ in the steady-state mode/conditions, which

also can be obtained from equations (II.34) and takes the form

$$m_X = \sum_{r=0}^{\infty} \frac{1}{r!} W_{0X}^{(r)}(0) f^{(r)}(t) + W_{0X}(0) m_n, \qquad \text{(II.39)}$$

where $W_{0X}(s)$ - transfer function of system (II.34) for the steady-state mode/conditions from input U to output X:

$$W_{0X}(s) = \frac{E_1(s) E_2(s) H_2(s)}{E_1(s) E_2(s) E_3(s) + k_0 H_1(s) H_2(s) H_3(s)}. \qquad \text{(II.40)}$$

Since useful signal f(t) is frequently polynomial, and system possesses astaticism of the necessary order, then series/rows in formulas (II.36) (II.37) (II.39) are broken and contain a finite number of terms.

Assuming also $k_1$ known and by constant, from linear stochastic equations (II.35) in the steady-state mode/conditions we compute the dispersions of the variables X, Y, using the formulas of the statistical dynamics of the stationary linear systems

$$D_X = \int_{-\infty}^{\infty} |W_{1X}(j\omega)|^2 S_n(\omega)\, d\omega; \qquad \text{(II.41)}$$

$$D_Y = \int_{-\infty}^{\infty} |W_{1Y}(j\omega)|^2 S_n(\omega)\, d\omega, \qquad \text{(II.42)}$$

where

$$W_{1Y}(j\omega) = \frac{k_1 H_1(j\omega) H_2(j\omega) E_3(j\omega)}{E_1(j\omega) E_2(j\omega) E_3(j\omega) + k_1 H_1(j\omega) H_2(j\omega) H_3(j\omega)}$$

$$W_{1X}(j\omega) = \frac{E_1(j\omega) E_3(j\omega) H_2(j\omega)}{E_1(j\omega) E_2(j\omega) E_3(j\omega) + k_1 H_1(j\omega) H_2(j\omega) H_3(j\omega)}.$$

Page 34.

The frequency characteristics of the physically possible stable system, described by ordinary differential equations, is the rational-linear function of frequency $\omega$. Spectral density $S_n(\omega)$ of stationary random process also can be represented or approximated in the form of the rational-linear function of frequency $\omega$. Consequently, integrands in formulas (II.41) and (II.42) can be represented in the form of rational-linear functions. For example, formula (II.41) takes the form

$$D_X = 2\pi I_n; \qquad \text{(II.43)}$$

$$I_n = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{g(j\omega)}{\eta(j\omega)\eta(-j\omega)}\,d\omega,$$

where

$$\eta_n(X) = a_0X^n + a_1X^{n-1} + \ldots + a_n;$$
$$g_n(X) = b_0X^{2n-2} + b_1X^{2n-4} + \ldots + b_{n-1}.$$

For integrals $I_n$ are comprised the tables, with the help of which these values are expressed as the coefficients of functions $g_n(j\omega)$ and $h_n(j\omega)$ (appendix 5). Equalities (II.39) (II.41) together with the expressions for $k_0 = k_0(m_X, D_X)$, $k_1 = k_1(m_X, D_X)$ are the equations, which mutually connect values $m_X$, $D_X$, $k_0$, $k_1$. Solving them together in any numerical manner, we determine $m_X$, $D_X$ and $k_0$, $k_1$. After this, after using formulas (II.36) (II.37) and (II.42), we determine $m_Y$, $m$, and $D_Y$ and, therefore, $D_\epsilon$, since in this case

$\epsilon^{\circ}=-y^{\circ}$, and $D_{\epsilon}=D_{Y}$.

Equations (II.39) and (II.41) can be solved by successive approximations or graphically. During the use/application of a method of successive approximations should be assigned the values of coefficients of $k_0$, $k_1$, also, according to formulas (II.39) (II.41) determined $m_x, D_x$ in the first approximation, then used the formulas for $k_0=k_0(m_x, D_x), k_1=k_1(m_x, D_x)$ for computing the new values of $k_0$ and $k_1$, after which the procedure of calculations is repeated. As a result they are calculated in second approximation/approach $m_x$ and $D_x$. Computations are finished, when two successive approximations coincide with an accuracy to errors in the calculations. Calçulation formulas easily are generalized to the case of system with the further inputs and the interferences. Let us consider this generalization based on example.

Page 35.

Example 1. To determine mathematical expectation and variance of error the servo fixed system under the effect of the stationary uncorrelated random disturbances at two inputs and upon consideration of the limitation (saturation) of the amplifier of control signal, if equations take the form

$$\left.\begin{aligned}(T_1s+1)sY&=aF(X)+n_2;\\ X&=k(Ts+1)(f+n_1-Y).\end{aligned}\right\}\qquad (II.44)$$

where $f=b_0+bt$; $b_0$ and $b$ - constant coefficients;

$n_1$ and $n_2$ - stationary random functions of time, which have

$$m_{n_1}=0;\ m_{n_2}=0;$$

$$S_{n_1}(\omega)=\frac{S_1}{(T_1\omega^2+1)(T\omega^2+1)};\ S_{n_2}(\omega)=\frac{S_2}{T\omega^2+1};\ S_1=\text{const};\ S_2=\text{const}.$$

The structural scheme of system is depicted in Fig. II.5, and II.6 the characteristic of nonlinearity F(X) is given.

Let us fulfill the statistical linearization of nonlinearity $F(X)=k_0m_X+k_1X^0$ and will record instead of equation (II.44) two systems of equations

$$\left.\begin{aligned}(T_1s+1)\,sm_Y&=ak_0m_X;\\ m_X&=k(Ts+1)(f-m_Y);\end{aligned}\right\}\qquad\text{(II.45)}$$

$$\left.\begin{aligned}(T_1s+1)\,sY^0&=ak_1X^0+n_2;\\ X^0&=k(Ts+1)(n_1-Y^0),\end{aligned}\right\}\qquad\text{(II.46)}$$

where

$$k_0=\frac{h}{m_X}\left\{\left(1+\frac{m_X}{C}\right)\Phi\left(\frac{C+m_X}{\sqrt{D_X}}\right)-\left(1-\frac{m_X}{C}\right)\Phi\left(\frac{C-m_X}{\sqrt{D_X}}\right)+\right.$$
$$\left.+\sqrt{\frac{D_X}{2\pi}}\,\frac{1}{C}\left[e^{-\frac{1}{2}\left(\frac{C+m_X}{\sqrt{D_X}}\right)^2}-e^{-\frac{1}{2}\left(\frac{C-m_X}{\sqrt{D_X}}\right)^2}\right]\right\};\qquad\text{(II.47)}$$

$$k_1=\frac{h}{C}\left[\Phi\left(\frac{C+m_X}{\sqrt{D_X}}\right)+\Phi\left(\frac{C-m_X}{\sqrt{D_X}}\right)\right].\qquad\text{(II.48)}$$

From equations (II.45) for the steady-state mode/conditions we

obtain

$$m_X = \frac{b}{ak_e};\qquad \text{(II.49)}$$

$$m_Y = 1 - \frac{b}{akk_e};\; m_e = \frac{b}{akk_e}.\qquad \text{(II.50)}$$

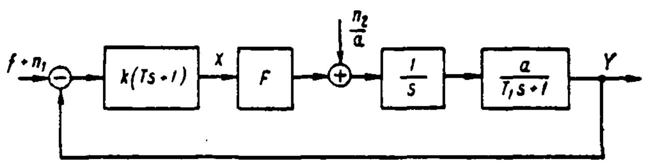



Fig. 5.

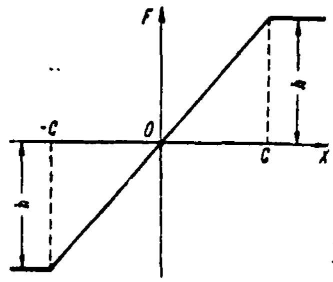



Fig. II.6.

Fig. II.5. Stabilization system.

Fig. II.6. Nonlinear characteristic.

Page 36.

From equations (II.46) for the steady-state mode/conditions we obtain the formulas, which determine $D_X$ and $D_\varepsilon = D_Y$. These formulas are the generalization of formulas (II.41) and (II.42) to the case of two inputs in the system:

$$D_X = \int_{-\infty}^{\infty} |W_{1X}(j\omega)|^2 S_{n_1}(\omega)\, d\omega + \int_{-\infty}^{\infty} |W'_{1X}(j\omega)|^2 S_{n_2}(\omega)\, d\omega; \quad (II.51)$$

$$D_Y = \int_{-\infty}^{\infty} |W_{1Y}(j\omega)|^2 S_{n_1}(\omega)\, d\omega + \int_{-\infty}^{\infty} |W'_{1Y}(j\omega)|^2 S_{n_2}(\omega)\, d\omega, \quad (II.52)$$

where the frequency characteristics, entering the formulas, take the following form:

$$
\left.\begin{aligned}
W_{1X}(j\omega) &= \frac{[T_1(j\omega)^2 + j\omega][Tj\omega + 1]k}{T_1(j\omega)^2 + (aTkk_1 + 1)j\omega + akk_1};\\
W'_{1X}(j\omega) &= \frac{k(Tj\omega + 1)}{T_1(j\omega)^2 + (aTkk_1 + 1)j\omega + akk_1};\\
W_{1Y}(j\omega) &= \frac{akk_1(Tj\omega + 1)}{T_1(j\omega)^2 + (aTkk_1 + 1)j\omega + akk_1};\\
W'_{1Y}(j\omega) &= \frac{1}{T_1(j\omega)^2 + (aTkk_1 + 1)j\omega + akk_1}.
\end{aligned}\right\} \quad (II.53)
$$

Calculating integrals in formulas (II.51) (II.52) at the preset above spectral densities of interferences we obtain

$$
D_X = \frac{(S_1 + S_2)\pi kk_1}{(1 + akk_1T)a}; \quad (II.54)
$$

$$
D_Y = \frac{\pi S_2(T + T_1 + akk_1T^2)}{akk_1(T + T_1 + 2T^2akk_1)(1 + akk_1T)} + \frac{\pi S_1akk_1(2T + mkk_1TT_1)}{(2T + akk_1TT_1)[1 + akk_1(T + T_1)] - akk_1T^2}. \quad (II.55)
$$

Let us execute concrete/specific/actual calculation with following data:

$$
a = 1;\ h = C = 0{,}1;\ T = 0{,}8 \text{ s } T_1 = 0.1 \text{ s};\ k = 2;
$$
$$
b = 0{,}05;\ S_1 = 4 \cdot 10^{-3};\ S_2 = 10 \cdot 10^{-3}.
$$

Substituting these values in equations (II.49) and (II.54), we obtain

$$
\left.\begin{aligned}
m_Xk_0(m_X, D_X) &= 0{,}05;\\
D_X &= \frac{0{,}088k_1(m_X, D_X)}{1 + 1{,}6k_1(m_X, D_X)},
\end{aligned}\right\} \quad (II.56)
$$

but coefficients $k_0$ and $k_1$ are preset by formulas (II.47) and

(II.48). Let us solve equations (II.56) by graphic. For this let us designate the right side of the first equation through $z_1=0.05$, and left - through $z_1 = m_x k_0(m_x; D_x)$. Let us construct these dependences in function $m_x$, after using formula (II.47) (Fig. II.7a), accepting $D_x$ for the parameter. Let us plot the points of intersection of these dependences in Fig. II.7b in coordinates $m_x, D_x$ and we draw curve 1. For the points of this curve we further compute the right side of second equation (II.56) during the use of formula (II.48) and we depict obtained dependence $D_x(m_x)$ on the graph as curve 2 (Fig. II.7b).

Page 37.

The point of intersection of curves 1 and 2 determines values $m_x = 0{,}103$; $D_x = 0{,}02$, which satisfy equations (II.55). In this case $k_0=0.494$, and $k_1=0.424$. Substituting $k_0$ and $k_1$ into formula (II.55), we will obtain $D_Y = D_1 = 0{,}012$.

The calculation procedure presented applies to the multidimensional stable fixed systems, which contain several nonlinear elements/cells. In this case only increases a number of together decided algebraic and transcendental equations of type (II.39) and (II.41) in accordance with an increase in the number of nonlinearity.

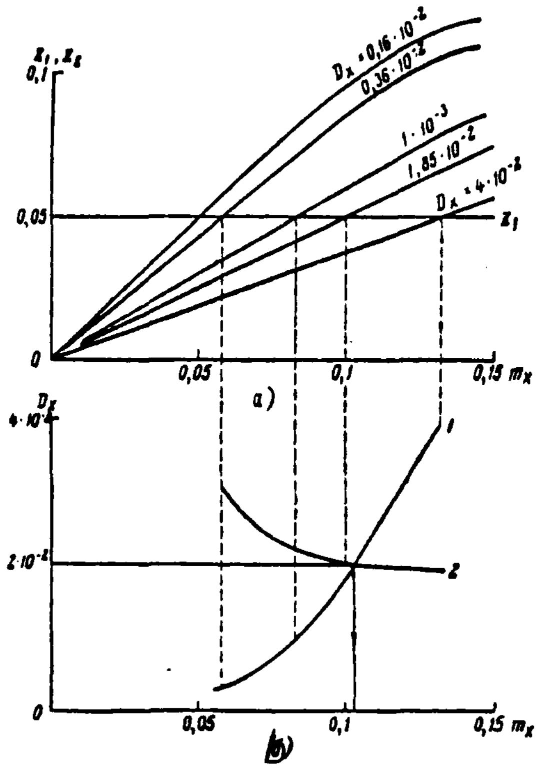



Fig. II.7. The graphical solution of system of equations.

Page 38.

3. Calculation of the accuracy of the nonlinear stationary discrete/digital control system.

The dynamics of discrete/digital system, as is known, is described by the functions of discrete/digital argument [76, 104]. System itself, as a rule, contains the elements of discrete/digital

operation and continuous parts, and nonlinearity can be both in the discrete/digital and in the continuous parts. For the presentation of the entity of task and simplification in the calculations let us consider one-dimensional fixed system with one odd nonlinearity in the continuous part, in which is absent self-vibrating mode/conditions. In the sufficiently general case the difference equations of this system for the continuous moments/torques of time $t = kT_n + \nu$ take the form

$$\left.\begin{aligned} E(\nabla)\, y(\kappa, \nu) &= H(\nabla)\, F[x(\kappa, \nu)];\\ F_{Д1}(\nabla)\, x(\kappa, \nu) &= H_{Д1}(\nabla)\,[U(\kappa) - y_1(\kappa, \nu)];\\ F_{Д2}(\nabla)\, y_1(\kappa, \nu) &= H_{Д2}(\nabla)\, y(\kappa, \nu), \end{aligned}\right\} \qquad (II.57)$$

where $\nabla$ - translation operator, for which $\nabla y(\kappa)=y(\kappa+1)=y(\kappa)=\nabla y(\kappa)$; $\kappa = t/T_n$; $T_n$ - step/pitch of discreteness; $0\leq\nu\leq1$; $E(\nabla)$, $H(\nabla)$ - the polynomials relative to operator $\nabla$ with the constant coefficients, which characterize the continuous part of the system; $F_{Д1}(\nabla)$, $H_{Д1}(\nabla)$, $F_{Д2}(\nabla)$, $H_{Д2}(\nabla)$ - polynomials with the constant coefficients, the characteristic discrete/digital parts of the system;

F - nonlinear characteristic. The structural scheme of system is depicted in Fig. II.8, where is marked $\psi(z) = H(z)/F(z)$ - the transfer function of linear part relative to discrete/digital argument $z = e^{sT_n}$;

$$\psi_{Д1}(z) = \frac{H_{Д1}(z)}{F_{Д1}(z)};$$

$\psi_{Д2}(z) = \dfrac{H_{Д2}(z)}{F_{Д2}(z)}$ - the transfer functions of the first and second discrete/digital parts of the system. At the input of system operate the useful nonrandom signal f(t) and the additive stationary random

noise n(t), so that for the particular moments of time let us record

$$U(\kappa) = f(\kappa) + n(\kappa),\ \kappa = 0, 1, \ldots \qquad \text{(II.58)}$$

Let the ideal operation L, which characterizes desired output signal $y_T(\kappa)$, take the form

$$y_T(\kappa) = \alpha_0 f^0(\kappa) + \alpha_1 f(\kappa) + \ldots + \alpha_N f^{(N)}(\kappa). \qquad \text{(II.59)}$$

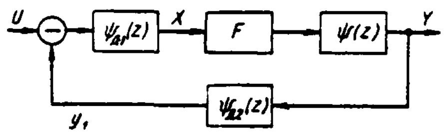


Fig. 11.8. Structural diagram of discrete nonlinear system.

Instantaneous error is expressed by the dependence

Page 39.

$$\varepsilon(\kappa, \nu) = y_T(\kappa, \nu) - y(\kappa, \nu). \qquad \text{(II.60)}$$

Applying the statistical linearization of odd nonlinearity, let us record

$$F(X) = k_0(m_X, D_X)\, m_X + k_1(m_X, D_X)\, X^0, \qquad \text{(II.61)}$$

where

$$\left.\begin{aligned} k_0 &= k_0(m_X, D_X); \\ k_1 &= k_1(m_X, D_X). \end{aligned}\right\} \qquad \text{(II.62)}$$

After this replacement of nonlinearity discrete/digital system (II.57) can be formally described by two systems of linear difference equations with respect to the average/mean and random signals. For the steady-state mode/conditions in the fixed system, which possesses the necessary level of astaticism, values $m_X$ and $D_X$ are constants. In this case the discrete/digital system in question is characterized

by stationary transfer functions. As a result for the steady-state mode/conditions we will obtain

$$m_X(\kappa)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\psi_{0X}^{(r)}(1)f^{(r)}(\kappa)+m_n(\kappa)\psi_{0X}(1); \quad \text{(II.63)}$$

$$m_\epsilon(\kappa)=\sum_{r=0}^{\infty}c_rf^{(r)}(\kappa)-m_n\psi_{0Y}(1). \quad \text{(II.64)}$$

In these expressions $c_r$ - error coefficients, which in this case are calculated from the formulas

$$\left.\begin{aligned} c_r&=\alpha_r-\frac{(-1)^r}{r!}\psi_{0Y}^{(r)}(1),\ r=0,1,\ldots,n,\\ c_r&=-\frac{(-1)^r}{r!}\psi_{0Y}^{(r)}(1),\ r=n+1,\ldots,\end{aligned}\right\} \quad \text{(II.65)}$$

where $\psi_{0X}(z)$ and $\psi_{0Y}(z)$ - transfer functions of discrete/digital system from input U to outputs X and Y respectively from the mathematical expectation, defined as

$$\psi_{0X}(z)=\frac{\psi_{Д1}(z)}{1+k_0(m_X,D_X)\psi_{Д1}(z)\psi_{Д2}(z)\psi(z)}; \quad \text{(II.66)}$$

$$\psi_{0Y}(z)=\frac{k_0(m_X,D_X)\psi_{Д1}(z)\psi(z)}{1+k_0(m_X,D_X)\psi_{Д1}(z)\psi_{Д2}(z)\psi(z)}. \quad \text{(II.67)}$$

Page 40.

For determining the dispersions $D_X$ and $D_\epsilon=D_Y$ we will use the linearized system of difference equations for random components. Let us determine transfer functions $\psi_{1X}(z)$ and $\psi_{1Y}(z)$ from input U to outputs X and Y respectively from them in the steady-state

mode/conditions:

$$\psi_{1X}(z) = \frac{\psi_{Д1}(z)}{1 + k_1(m_X, D_X)\psi_{Д1}(z)\psi_{Д2}(z)\psi(z)}; \qquad (II.68)$$

$$\psi_{1Y}(z) = \frac{k_1(m_X, D_X)\psi_{Д1}(z)\psi(z)}{1 + k_1(m_X, D_X)\psi_{Д1}(z)\psi_{Д2}(z)\psi(z)}. \qquad (II.69)$$

After this should be computed the integrals, which determine the dispersions of variable/alternating linear stationary discrete/digital stable systems in steady-state mode/conditions [76, 104]:

$$D_X(\kappa) = \int_{-\infty}^{\infty} \left|\psi_{1X}\left(\frac{1+j\sigma}{1-j\sigma}\right)\right|^2 \rho_n^d(j\sigma)\frac{2}{1+\sigma^2}\,d\sigma; \qquad (II.70)$$

$$D_\varepsilon(\kappa) = \int_{-\infty}^{\infty} \left|\psi_{1Y}\left(\frac{1+j\sigma}{1-j\sigma}\right)\right|^2 \rho_n^d(j\sigma)\frac{2}{1+\sigma^2}\,d\sigma, \qquad (II.71)$$

where

$$\rho_n^d(j\sigma) = \frac{1}{T_n} S_n^d\left(\frac{1}{iT_n}\ln\frac{1+j\sigma}{1-j\sigma}\right);$$

$S_n^d(\omega)$ - the spectral density of the discrete/digital stationary random process $n^*(\kappa)$.

Integrals in formulas (II.70) and (II.71) are calculated according to the tables, given in the application/appendix.

Formulas (II.63) and (II.70) are equations for determination $m_X(\kappa)$ and $D_X(\kappa)$. Connecting to them equations for $k_0(m_X, D_X)$, $k_1(m_X, D_X)$ and solving them together by successive approximations or

graphically, we determine $m_x, D_x, k_0, k_1$. Then according to formulas (II.64) and (II.71) we compute values $m_t$ and $D_t$ for the steady-state mode/conditions for the particular moments. The procedure of the calculation of the accuracy of the simplest discrete/digital system presented is analogous to the calculation of the accuracy of the simplest discrete/digital system it is analogous to the calculation of continuous systems; therefore it easily is generalized to the multidimensional systems, which have several nonlinearity.

Example 1. The stabilization system of gyro horizon, which contains the corrected gyroscope with the transfer function k/S, nonlinear compensator of the type of ideal relay with characteristic F(x)=h sign x and pulse element/cell with δ-characteristic, is located under the interaction of the stationary random disturbance n of the type of white noise with the constant mathematical expectation. The structural scheme of system is depicted in Fig. II.9.

Page 41.

It is necessary to determine mathematical expectation and variance of error of the stabilization of vertical line in the steady-state mode/conditions.

For the composition of the equations of discrete/digital system let us determine the before discrete/digital transfer function of the linear part of the system with δ-pulse discrete element, after using z-conversion of the output signal of the linear part of the system. With the transfer function of the linear part k/S according to the table of z-conversion we find

$$\Psi(z)=\frac{kz}{z-1}. \qquad (II.72)$$

This transfer function also can be determined, if to compose the difference equation of the continuous part of the system. The difference equation of closed system now can be recorded in the form

$$\left.\begin{aligned} Y(\kappa)&=\Psi(\nabla)F[X(\kappa)]\\ X(\kappa)&=n(\kappa)-Y(\kappa);\end{aligned}\right\} \qquad (II.73)$$

Is realized the statistical linearization of nonlinearity F(X):

$$F(X)=k_0(m_X,D_X)m_X+k_1(m_X,D_X)X^0, \qquad (II.74)$$

where

$$k_0=\frac{2h}{m_X}\Phi\left(\frac{m_X}{\sqrt{D_X}}\right);\ k_1=\frac{2h}{\sqrt{2\pi D_X}}e^{-\frac{m_X^2}{2D_X}}$$

are calculated for the particular moments of time $t=\kappa T_n$. After linearization the system of equations (II.73) is converted into two connected systems

$$\left.\begin{aligned} m_Y(\kappa)&=\Psi(\nabla)k_0m_X(\kappa);\\ m_X(\kappa)&=m_n(\kappa)-m_Y(\kappa);\end{aligned}\right\} \qquad (II.75)$$

$$\left.\begin{aligned} Y^0(\kappa)&=\Psi(\nabla)k_1X^0(\kappa);\\ X^0(\kappa)&=n^0(\kappa)-Y^0(\kappa).\end{aligned}\right\} \qquad (II.76)$$

In steady-state mode/conditions $m_x$ and $D_x$ they are constant values at the particular moments of time $t=\kappa T_n$. in question Therefore systems of equations (II.75) and (II.76) for this mode/conditions are stationary. Using them, let us determine the discrete/digital transfer functions of closed systems for outputs $m_X$, $m_Y$, $X^0$, $Y^0$.

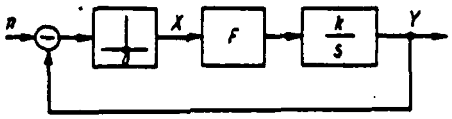


Fig. 11.9. Structural diagram of nonlinear system.

Page 42.

These transfer functions in this case take the form

$$\left.\begin{aligned}
\Psi_{0X}(z) &= \frac{z-1}{[1+kk_0(m_X, D_X)]z-1};\\
\Psi_{1X}(z) &= \frac{z-1}{[1+kk_1(m_X, D_X)]z-1};\\
\Psi_{0Y}(z) &= \frac{kk_0(m_X, D_X)z}{[1+kk_0(m_X, D_X)]z-1},\\
\Psi_{1Y}(z) &= \frac{kk_1(m_X, D_X)z}{[1+kk_1(m_X, D_X)]z-1}.
\end{aligned}\right\} \qquad (II.77)$$

After using formulas (II.63) and (II>64) and bearing in mind that in this case the desired value of output coordinate $Y_T=0$ and f=0, we determine $m_x$ and $m_\varepsilon=-m_Y$:

$$m_X = m_n \Psi_{0X}(1) = 0;\ m_\varepsilon = -m_n. \qquad (II.78)$$

Thus, coefficient $k_1$ depends in this case only on value $D_x$. For determining the value $D_x$ we will use formula (II.70), having preliminarily computed the spectral density of the discrete/digital stationary white noise $n^*(\kappa)$. Spectral density $S_n^d$ of this stationary

random sequence of uncorrelated pulses is equal to

$$S_n^d(\omega) = \frac{T_n D}{2\pi},$$

where D - dispersion of impulse/momentum/pulse.

Passing to variable $z = e^{i\omega T_n}$, let us record spectral density in the form

$$\sigma_n^d(z) = \frac{1}{T_n} S_n^d\left(\frac{1}{iT_n}\ln z\right) = \frac{D}{2\pi}.$$

After passing now to by the variable w according to formula z=(1+w)/(1-w), we will obtain

$$\rho_n^d(w) = \frac{D}{2\pi}.$$

In the formula for $\Psi_{1X}$ it is realized also replacement by the variable z on the variable w:

$$\Psi_{1X}\left(\frac{1+w}{1-w}\right) = \frac{2w}{[2+kk_1]\,w + kk_1}. \qquad (II.79)$$

We substitute expression (II.79) in formula (II.70), after replacing $w=i\sigma$:

$$D_X = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{2j\sigma}{[2+kk_1(D_X)]\,j\sigma + kk_1(D_X)}\right|^2 \frac{2D}{1+\sigma^2}\,d\sigma. \qquad (II.80)$$

Calculating integral in formula (II.80) according to tables, we obtain

$$D_X = \frac{2D}{[2+kk_1(D_X)][1+kk_1(D_X)]};\quad k_1 = \frac{2h}{\sqrt{2\pi D_X}}. \qquad (II.81)$$

Page 43.

From equations (II.81) we determine

$$\sqrt{D_X} = -\frac{3kh}{2\sqrt{2\pi}} + \sqrt{D + \frac{k^2h^2}{8\pi}}. \qquad (II.82)$$

Now according to formula (II.71) we determine the variance of error of stabilization, which after the replacement of variables takes the form

$$D_Y = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{kk_1 i\omega}{(2+kk_1)\,i\omega + kk_1}\right|^2 \frac{2D}{1+\omega^2}\,d\omega. \qquad (II.83)$$

Calculating integral, we will obtain

$$D_Y = D\frac{kk_1}{kk_1+2}. \qquad (II.84)$$

4. Calculation of the accuracy of control processes in the transient nonlinear systems by the method of canonical expansions.

A large number of objects of control is characterized by transient equations, therefore, processes in these objects are transient. Furthermore, during the study of transient process in the fixed system also we come to the transient task. For solving the whole series of tasks during the research of control processes in the transient nonlinear systems, on which act random disturbances, successfully are applied the methods of canonical expansions and the

Page 43.

From equations (II.81) we determine

$$\sqrt{D_X} = -\frac{3kh}{2\sqrt{2\pi}} + \sqrt{D + \frac{k^2h^2}{8\pi}}. \qquad \text{(II.82)}$$

Now according to formula (II.71) we determine the variance of error of stabilization, which after the replacement of variables takes the form

$$D_Y = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{kk_1 i\omega}{(2+kk_1)\,i\omega + kk_1}\right|^2 \frac{2D}{1+\omega^2}\,d\omega. \qquad \text{(II.83)}$$

Calculating integral, we will obtain

$$D_Y = D\frac{kk_1}{kk_1+2}. \qquad \text{(II.84)}$$

4. Calculation of the accuracy of control processes in the transient nonlinear systems by the method of canonical expansions.

A large number of objects of control is characterized by transient equations, therefore, processes in these objects are transient. Furthermore, during the study of transient process in the fixed system also we come to the transient task. For solving the whole series of tasks during the research of control processes in the transient nonlinear systems, on which act random disturbances, successfully are applied the methods of canonical expansions and the

integrations of the equations of probabilistic moments/torques, based on the statistical linearization of nonlinearity. In this paragraph let us consider the method of canonical expansions.

In the sufficiently general case in the presence of one one-dimensional nonlinearity the behavior of continuous transient nonlinear system is characterized by the following equation relative to output by the variable Y(t) with the preset input function U(t):

$$E(t, s)Y + R(t, s)F(Y) = H(t, s)U, \qquad (\text{II}.85)$$

where

$$E(t, s) = \sum_{r=1}^{n} a_r(t)s^r; \; R(t, s) = \sum_{r=1}^{l} c_r(t)s^r;$$

$$H(t, s) = \sum_{r=1}^{m} b_r(t)s^r;$$

F(Y) - nonlinear function.

Page 44.

After the statistical linearization of nonlinearity equation (II.85) can be rewritten in the form of the system of two connected equations - nonlinear (for the mathematical expectation) and linear (for random component):

$$E(t, s)m_Y + R(t, s)F_0(m_Y, D_Y) = H(t, s)m_U; \qquad (\text{II}.86)$$

$$E(t, s)Y^0 + R(t, s)k_1(m_Y, D_Y)Y^0 = H(t, s)U^0, \qquad (\text{II}.87)$$

where $F_0(m_Y, D_Y)$; $k_1(m_Y, D_Y)$ have concrete/specific/actual analytical

expression depending on the form of nonlinearity.

Let us assume that the input signal U(t)=f(t)+n(t) is represented by any canonical expansion [34, 76]

$$U(t) = m_U(t) + \sum_{j=1}^{N} V_j u_j(t), \qquad \text{(II.88)}$$

where $m_U(t) = f(t) + m_n(t)$ - mathematical expectation of function;

$u_j(t)$ - coordinate functions;

$V_j$ - random not connected coefficients, which have dispersions $D_j$ and equal to zero mathematical expectations.

Equation (II.86) should be integrated under preset initial conditions $t = 0,\ m_Y(0), \dots, m_Y^{(n-1)}(0)$. The solution of equation (II.87) is found out in the form of canonical expansion with the same random coefficients and new unknown coordinate functions

$$Y^0(t) = \sum_{j=1}^{N} V_j y_j(t).$$

Substituting this expression and

$$U^0(t) = \sum_{j=1}^{N} V_j u_j(t)$$

in equation (II.87) and using formally the principle of superposition, we obtain the equation

$$E(t, s) Y_j + R(t, s) k_1(m_Y, D_Y) y_j = H(t, s) u_j,\ j = 1, \dots, N. \qquad \text{(II.89)}$$

for determining the coordinate function of number j.

These
equations also must be integrated under the specified initial conditions, if initial conditions for random dispersions $D_{Y_0}, \ldots, D_{Y_0^{(n-1)}}$. component in the form are preset

Page 45.

Thus, for instance, initial conditions can be selected not zero only for the coordinate function $y_1(t)$:

$$t = 0,\ y_1^{(r)} \sqrt{\frac{D_{Y_0^{(r)}}}{D_1}},\ r = 0, 1, \ldots, n-1,$$

and equations for coordinate functions $y_2, \ldots, y_N$ are integrated under the zero initial conditions. Obtained equations (II.86) and (II.89) must be supplemented with formulas for functions $F_0 = F_0(m_Y, D_Y)$ and $k_1 = k_1(m_Y, D_Y)$, and also with expression for dispersion $D_Y$:

$$D_Y = \sum_{j=1}^{N} D_j y_j(t) \cdot \bar{y}_j(t), \qquad \text{(II.90)}$$

where $\bar{y}_j(t)$ - complexly adjoint function.

Thus, the problem of determining of mathematical expectation and dispersion by the variable Y, which satisfies equation (II.85), is solved by the combined integration of one equation for mathematical expectation (II.86), N of equations for coordinate functions (II.89) during the use of formula (II.90) and two formulas for $F_0$ and $k_1$.

If the desired value of output function $y_T(t) = Lm_U$, is preset mathematical expectation and variance of error are determined from the formulas

$$m_\varepsilon = y_T(t) - m_Y(t);\ D_\varepsilon = D_Y. \qquad \text{(II.91)}$$

As a result can be calculated the mean square of the error:

$$\alpha_2 = m_\varepsilon^2 + D_\varepsilon. \qquad \text{(II.92)}$$

The described algorithm of the solution of problem virtually can be realized in the digital computer. It should be noted that during the use/application of the algorithm in question can be calculated all correlation functions of variables. For example, instead of formula (II.90) there can be used more general/common/total expression for the correlation function

$$K_Y(t, t_1) = \sum_{j=1}^{N} D_j y_j(t) \bar{y}_j(t_1). \qquad \text{(II.93)}$$

The procedure of research of the simplest system presented with one nonlinearity applies to continuous and discrete/digital systems with several nonlinearity and inputs. In this case increases the number of final formulas, from which are calculated statistical amplification factors. Furthermore, it is necessary to additionally calculate the dispersions of variables at the inputs of nonlinearity according to the formulas of type (II.90). For the discrete/digital systems all differential equations must be replaced differential. The

practical use/application of a method of canonical expansions is expedient with a small number of members of the canonical sum, by which can be represented the random disturbances.

Page 46.

Example 1. Nonlinear stationary stabilization system has equations

$$E(s)Y = H(s)F(X);\ X = U - Y, \qquad (II.94)$$

where E(s) and H(s) - polynomials with the constant coefficients; F - odd nonlinear characteristic of an arbitrary form, for example, of the type of clipping (Fig. II.6). Stationary random noise of the type of the white noise U(t) with mathematical expectation $m_U$ operates on the system. The structural scheme of system is represented in Fig. II.10. It is necessary to determine mathematical expectation and dispersion of the output variable Y(t) in the transient mode/conditions under the initial condition t=0, $m_Y(0) = y_0$, $D_Y(0) = D_0$. All remaining initial conditions are equal to zero.

After using the statistical linearization of nonlinearity F, let us record instead of equation (11.94) the following equations for the mathematical expectations and central components:

$$E(s)\,m_Y = H(s)\,k_0(m_X,\,D_X)\,m_X; \qquad (II.95)$$

$$m_X = m_U - m_Y;$$

$$\left.\begin{aligned} E(s)\,Y^0 &= H(s)\,k_1(m_X,\,D_X)\,X^0; \\ X^0 &= U^0 - Y^0. \end{aligned}\right\} \qquad (II.96)$$

The integration of equation (II.95) should be conducted under the initial condition $t=0,\ m_Y=y_0$; remaining initial conditions they are equal to zero. Coefficients $k_0$ and $k_1$ in this case are determined from the formulas

$$k_0 = \frac{h}{m_X}\left\{\frac{C+m_X}{d}\,\Phi\left(\frac{C+m_X}{\sqrt{D_X}}\right) - \frac{C-m_X}{d}\,\Phi\left(\frac{C-m_X}{\sqrt{D_X}}\right) + \right.$$
$$\left. + \frac{\sqrt{D_X}}{\sqrt{2\pi}}\left[\exp\left(-\frac{1}{2}\left(\frac{C+m_X}{\sqrt{D_X}}\right)^2\right) - \exp\left(-\frac{1}{2}\left(\frac{C-m_X}{\sqrt{D_X}}\right)^2\right)\right]\right\};$$
$$k_1 = \frac{h}{d}\left[\Phi\left(\frac{C+m_X}{\sqrt{D_X}}\right) + \Phi\left(\frac{C-m_X}{\sqrt{D_X}}\right)\right]. \qquad (II.97)$$

The random stationary perturbation $U^0(t)$ let us present to approximation/approach in the form of canonical expansion with a limited number of members of the sum:

$$U^0(t) \approx \sum_{\nu=1}^{n}\left[V_\nu \cos\omega_\nu t + W_\nu \sin\omega_\nu t\right], \qquad (II.98)$$

where n - number of small intervals $\Delta\omega = \frac{\omega_n}{n}$;

- the maximum frequency of the passband of the system in question, which is approximately/exemplarily equal to $\omega_n \approx \frac{3}{T}$, and T - transit time in the system.

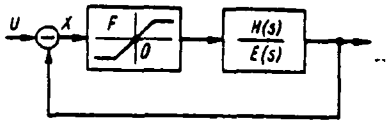



Fig. II.10. Nonlinear continuous stabilization system.

Page 47.

The random coefficients of expansion $V_\nu$, $W_\nu$ are subordinated to the conditions

$$M[V_\nu V_\mu] = M[W_\nu W_\mu] = \begin{cases} 0, & \nu \neq \mu \\ \dfrac{2\pi S_0 \omega_n}{n}, & \nu = \mu, \end{cases} \qquad \text{(II.99)}$$
$$M[V_\nu W_\mu] = 0, \ \nu, \mu = 1, \ldots, n,$$

where $S_0$ - the spectral density of the white noise $U^0(t)$.

As it was indicated above, all random variables, entering equations (II.96), must be represented also in the form of canonical expansions with the same random coefficients, but unknown coordinate functions:

$$\left.\begin{aligned} Y^0 &= \sum_{\nu=1}^{n} (V_\nu y'_\nu + W_\nu y''_\nu); \\ X^0 &= \sum_{\nu=1}^{n} (V_\nu x'_\nu + W_\nu x''_\nu). \end{aligned}\right\} \qquad \text{(II.100)}$$

For determining the unknown coordinate functions serve the equations, obtained from system (II.96):

$$\left.\begin{array}{l} E(s)\,y_\nu' = H(s)\,k_1(m_X, D_X)\,x_\nu';\ x_\nu' = \cos\omega_\nu t - y_\nu'; \\ E(s)\,y_\nu'' = H(s)\,k_1(m_X, D_X)\,x_\nu'';\ x_\nu'' = \sin\omega_\nu t - y_\nu'', \end{array}\right\} \quad \text{(II.101)}$$

During the integration of equations (II.101) let us take the following initial conditions: $t=0,\ y_\nu'(0)=\sqrt{\frac{D_Y}{2\pi S_0\omega_n}}$. Let us take as remaining initial conditions equal to zero.

Dispersiuns $D_Y$ and $D_X$ we determine from the formulas

$$\left.\begin{array}{l} D_X = \frac{2\pi S_0\omega_n}{n}\sum_{\nu=1}^{n}\left[x_\nu'^2(t) + x_\nu''^2(t)\right]; \\ D_Y = \frac{2\pi S_0\omega_n}{n}\sum_{\nu=1}^{n}\left[y_\nu'^2(t) + y_\nu''^2(t)\right]. \end{array}\right\} \quad \text{(II.102)}$$

As a result for determination $m_Y(t)$ and $D_Y(t)$ it is necessary to together integrate systems of equations (II.95) (II.101) upon consideration of formulas (II.97) and (II.102).

5. Calculation of the accuracy of control processes in the transient nonlinear systems by the method of integrating the momental equations.

Page 48.

Second method considered/examined here of the study of the transient nonlinear control systems under the effect of random

disturbances/perturbations is based on the representation of dynamic equations in the canonical form relative to the first derived phase coordinates with the additive white noises - disturbances/perturbations in the right sides:

$$\dot{Y}_\kappa = F_\kappa(t, Y_1, \ldots, Y_n) + V_\kappa(t),\ \kappa = 1, \ldots, n, \qquad \text{(II.103)}$$

where $F_\kappa$ - nonlinear functions of general view, into which enter linear terms; $V_\kappa(t)$ - random functions of the type of gaussian white noises. After the statistical linearization of nonlinearity $F_\kappa$ we obtain two systems of the linearized equations

$$m_{Y_\kappa} = F_{\kappa 0} + m_{V_\kappa},\ \kappa = 1, \ldots, n; \qquad \text{(II.104)}$$

$$\dot{Y}^0_\kappa = \sum_{r=1}^{n} k_{\kappa r} Y^0_r + V^0_\kappa,\ \kappa = 1, \ldots, n, \qquad \text{(II.105)}$$

where $F_{\kappa 0} = M(F_\kappa)$, $k_{\kappa r}$ - statistical amplification factors depending on the mathematical expectations and covariances of the connection/communication of phase variables.

The method of determining the instantaneous values of covariances of coordinates consists of the replacement stochastic equations (II.105) by the equations, which relate covariances $D_{ij} = \overline{M}[Y^0_i(t) Y^0_j(t)]$ of coordinates with the probabilistic moments/torques of external disturbances/perturbations. For obtaining these equations let us compute time derivative of $D_{ij}(t)$:

$$\dot{D}_{ij} = M[\dot{Y}^0_i Y^0_j] + M[Y^0_i \dot{Y}^0_j], \quad i, j = 1, \ldots, n. \qquad \text{(II.106)}$$

Using equations (II.105), we will obtain

$$\dot{D}_{ij} = \sum_{r=1}^{n} [k_{ir}D_{rj} + k_{jr}D_{ri}] + M\left[Y_j^0(t)V_i^0(t)\right] +$$
$$+ M\left[Y_i^0(t)V_j^0(t)\right], \quad i, j = 1, \ldots, n. \qquad \text{(II.107)}$$

Covariances $M[Y_i^0(t)V_j^0(t)]$, entering equations (II.107), are easily calculated from the formulas

$$M\left[Y_i^0(t)V_j^0(t)\right] = \sum_{l=1}^{n} \int_0^t g_{il}(t, \tau) M[V_j^0(t) \times$$
$$\times V_l^0(\tau)]\, d\tau, \quad i, j = 1, \ldots, n, \qquad \text{(II.108)}$$

where $g_{il}(t, \tau)$ - weight functions of system (II.105). Taking into account that $V_j^0(t)$ - white noises, for which $M[V_j^0(t)V_l^0(\tau)] = G_{jl}(t)\delta(t-\tau)$, $G_{jl}(t)$ - mutual intensities, and the functions

$$g_{il}(t, t) = \begin{cases} 1, & i = l \\ 0, & i \neq l, \end{cases}$$

we will obtain

$$M\left[Y_i^0(t)V_j^0(t)\right] = \frac{1}{2} G_{ij}. \qquad \text{(II.109)}$$

Page 49.

Consequently, equations (II.107) take the form

$$\dot{D}_{ij} = \sum_{r=1}^{n} [k_{ir}D_{rj} + k_{jr}D_{ri}] + G_{ij}, \qquad \text{(II.110)}$$
$$i, j = 1, \ldots, n.$$

It is necessary to take into account in equations (II.110) that $D_{ij}=D_{ji}$, therefore a number of independent equations (II.110) is equal to $\frac{n(n+1)}{2}$, where n - order of the reference system of equations. Equations (II.104) and (II.110) are the unknown system. Must be integrated them under the preset initial conditions t=0, $m_{Y_i}(0)$, $D_{ij}(0)$, $i, j=1, \ldots, n$. One should, however, take into account that these equations are connected through functions $F_{\kappa 0}$ and coefficients $k_{\kappa r}$, which depend on the mathematical expectations and covariances of the corresponding variables. Therefore to equations (II.104) and (II.110) it is necessary to connect formulas for $F_{\kappa 0}$ and $k_{\kappa r}$:

$$\left.\begin{array}{l} F_{\kappa 0}=F_{\kappa 0}(t, m_{Y_1}, \ldots, D_{nn}), \\ k_{\kappa r}=k_{\kappa r}(t, m_{Y_1}, \ldots, D_{nn}) \\ \kappa, r=1, \ldots, n. \end{array}\right\} \quad \text{(II.111)}$$

Thus, it is necessary to together integrate equations (II.104) (II.110) taking into account formulas (II.111), i.e., in all is obtained $\frac{n(n+3)}{2}$ differential first-order equations.

The methods of the study of dynamic stochastic systems, presented in this and in previous paragraphs, lead to the need for the combined single integration of the system of ordinary differential equations. The comparison of the space of computations during the use/application of a method of canonical expansions and method of the integration of momental equations shows that with (n+1)/2<N, where n - order of the reference system of equations, and

N - number of terms of the combined canonical expansion of disturbances/perturbations, the second method on the labor expense is more advantageous.

The accuracy of the work of system during the use/application of the method presented in this paragraph in any of the variables also can be evaluated according to formulas (II.91) (II.92), if is preset input useful and desired output signals.

Page 50.

The method of integrating the instantaneous values of covariances of connection/communication is applicable also to the discrete/digital systems of control, whose dynamics is characterized by difference equations in the normal form

$$\Delta Y_\kappa(q) = F_\kappa(q, Y_1, \ldots, Y_n) + V_\kappa(q), \qquad \text{(II.112)}$$

where $\Delta Y_\kappa(q)$ - first differences;

$q = \frac{t}{T_n}$;

$V_\kappa(q)$ - random white noises.

After the statistical linearization of nonlinear functions $F_\kappa(q, Y_1, \ldots, Y_n)$ we obtain the statistically linearized difference equations for the mathematical expectations and central components:

$$\Delta m_{Y_\kappa}(q) = F_{\kappa 0}(q, m_{Y_1}, \ldots, D_{nn}) + m_{V_\kappa}(q), \qquad \text{(II.113)}$$
$$\kappa = 1, \ldots, n$$
$$\Delta Y^0_\kappa(q) = \sum_{r=1}^{n} k_{\kappa r}(q, m_{Y_1}, \ldots, D_{nn}) Y^0_r(q) + V^0_\kappa(q), \qquad \text{(II.114)}$$
$$\kappa = 1, \ldots, n$$

where $D_{ji}(q)$ - covariances of the connection/communication of variables $Y^0_j(q)$ and $Y^0_i(q)$;

$F_{\kappa 0}$ - statistically averaged functions $F_\kappa$;

$k_{\kappa r}$ - statistical amplification factors.

Equations (II.114) serve for the composition of the difference equations, which determine covariances. These equations are composed by the path, analogous to the procedure of obtaining equations for the continuous systems, and take the form

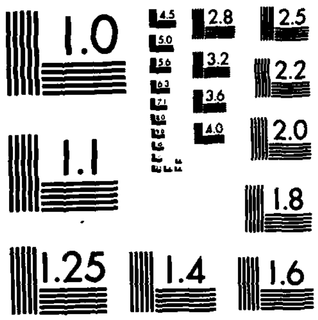
$$\Delta D_{ij}(q) = \sum_{r=1}^{n} [k_{ir}(q) D_{rj}(q) + k_{jr}(q) D_{ri}(q) +$$
$$+ k_{ir}(q) \sum_{\rho=1}^{n} k_{j\rho}(q) D_{r\rho}(q) + G_{ij}(q), \qquad \text{(II.115)}$$
$$i, j = 1, \ldots, n$$

where $G_{ij}(q)$ - mutual intensities of discrete/digital white noises $V^0_i(q)$ and $V^0_j(q)$.

Connecting to equations (II.113) (II.115) formula for $F_{\kappa 0}$ and

AD-A143 297 STATIC METHODS IN THE DESIGN OF NONLINEAR AUTOMATIC CONTROL SYSTEMS(U) FOREIGN TECHNOLOGY DIV WRIGHT-PATTERSON AFB OH N I ANDREYEV ET AL. 27 JUN 84 2/9

UNCLASSIFIED FTD-ID(RS)T-1734-83 F/G 12/1 NL


1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0

$k_{\kappa r}$ we obtain full/total/complete system of equations for the joint decision.

Page 51.

Being given the initial values of the first and second moments/torques for q=0 and solving these equations in the digital computer, we determine for the consecutive particular moments of time mathematical expectations and covariances of the connection/communication of all variables. For the evaluation/estimate of the accuracy of discrete/digital systems serve the same formulas (II.91) and (II.92).

Example 1. It is necessary to investigate the transient process of the establishment of mathematical expectation and variance of error of gyro horizon with the nonlinear correction of the interactions of stationary random disturbance/perturbation in the presence of random initial error with the preset mathematical expectations and the dispersion. The equations of gyro horizon take the form

$$\dot{Y} = F(X);\ X = U - Y, \qquad \text{(II.116)}$$

where U(t) - the stationary random process, which has $m_U = \mathrm{const}$ and U°(t) with a spectral density of $S_U(\omega) = \frac{D_0}{\pi} \cdot \frac{\alpha}{\alpha^2 + \omega^2}$ $F(X) = h\,\mathrm{sign}\,X$. The structural scheme of gyro horizon is given in Fig. II.11.

Let us reduce equations (II.116) to the system of normal equations with the white noises in the right side. For this we will use the expression of random function U°(t) through the white noise with the help of forming filter [76]. Let us represent rational-linear function $S_U(\omega)$ in the form

$$S_U(\omega)=\sqrt{\frac{D_0\alpha}{\pi}}\cdot\frac{1}{\alpha+j\omega}\sqrt{\frac{D_0\alpha}{\pi}}\cdot\frac{1}{\alpha-j\omega}. \qquad \text{(II.117)}$$

On the other hand, if there is forming filter [76] with the transfer function W(s), then spectral density $S_U(\omega)$ of random process at the output of this filter under the effect at the input of stationary white noise with the single spectral density and an intensity of $G=2\pi$ is equal to

$$S_U(\omega)=W(j\omega)\,W(-j\omega). \qquad \text{(II.118)}$$

On the basis of expressions (II.117) and (II.118) we conclude that the transfer function of the forming filter takes the form

$$W(s)=\sqrt{\frac{D_0\alpha}{\pi}}\cdot\frac{1}{\alpha+s}. \qquad \text{(II.119)}$$

Thus, the stationary random function U°(t) is connected with the white noise V°(t), which has spectral density $S_V=1$, with the equation

$$(s+\alpha)\,U^{\circ}=\sqrt{\frac{D_0\alpha}{\pi}}\,V^{\circ}. \qquad \text{(II.120)}$$

Consequently, equations (II.116) are replaced by the following system:

$$\left.\begin{aligned}&\dot{Y}=F(X);\ X=U-Y;\\&\dot{U}=-\alpha U+\alpha m_U+\sqrt{\frac{D_0\alpha}{\pi}}\,V^{\circ},\end{aligned}\right\} \qquad \text{(II.121)}$$

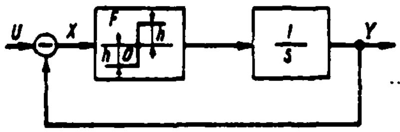



Fig. II.11. Structural scheme of gyro horizon.

Page 52.

Let us fulfill the statistical linearization of the function

$$F(X) = k_0 m_X + k_1 X^0, \qquad \text{(II.122)}$$

where

$$k_0 = \frac{2h}{m_X}\Phi\left(\frac{m_X}{\sqrt{D_X}}\right); \quad k_1 = \frac{2h}{\sqrt{2\pi D_X}}\, e^{-\frac{m_X^2}{2D_X}}.$$

Substituting expression (II.122) into equations (II.121) and separating/liberating mathematical expectations and random components, we will obtain the equations

$$\dot{m}_Y = k_0 m_X; \quad m_X = m_U - m_Y; \qquad \text{(II.123)}$$

$$\dot{Y}^0 = k_1 X^0; \quad X^0 = U^0 - Y^0; \quad \dot{U}^0 = -\alpha U^0 + \sqrt{\frac{D_0 \alpha}{\pi}}\, V^0. \qquad \text{(II.124)}$$

for them.

To equations (II.123) correspond the following equations for covariances, in which it is taken into consideration, that $U^0(t)$ - stationary random function $D_{UU} = D_0 = \text{const}$:

$$\dot{D}_{YY} = -2k_1 D_{YY} + 2k_1 D_{YU}; \quad \dot{D}_{YU} = -(k_1 + \alpha)\times$$
$$\times D_{YU} + k_1 D_0; \quad D_{XX} = D_0 + D_{YY} - 2D_{YU}. \qquad \text{(II.125)}$$

During the integration of equations (II.123) it is necessary to take into account the initial conditions t=0; $m_Y(0)=m_{Y_0}$, while during the integration of equations (II.124) - the initial conditions t=0; $D_Y(0)=D_{Y_0}$; $D_{YU}(0)=0$. Equations (II.123) (II.124) together with formulas (II.122) for $k_0$ and $k_1$ are closed system, and they integrate them together. As a result of integration are determined unknown dependences $m_Y(t)$ and $D_Y(t)$.

6. Calculation of systems with the random parameters.

The theoretically identical automatic control systems have the random parameters due to the tolerances at the production of the single elements of systems, and also in view of a change of their properties in the process of work. In the majority of the cases these random parameters can be considered random variables with the given average/mean values and dispersions.

The possible values of the parameters are located in the tolerance range. The random parameters as has already been spoken in Chapter 1, even in the system, linear relative to variables, enter multiplicatively. Therefore any system with the random parameters is stochastic nonlinear. The use/application of a method of statistical linearization is one of the approximate methods of the analysis of such systems.

Page 53.

Let us examine, for example, linear stabilization system with the random parameters:

$$E(s, a_i)Y = Q(s, b_j)U \quad (\text{II}.126)$$
$$(i = 1, 2, \ldots, \kappa;\ j = 1, 2, \ldots, l),$$

where U(t) - the random stationary interference, which has constant value of mathematical expectation $m_U$ and spectral density $S_U(\omega)$;

$E(s, a_i), Q(s, b_j)$ - polynomials relative to s with the random parameters

$$E(s, a_i) = c_0 s^{\kappa} + \ldots + a_{\kappa},\ Q(s, b_j) = b_0 s^l + \ldots + b_l.$$

Random parameters $a_1, \ldots, a_{\kappa}, b_1, \ldots, b_l$, are not connected and with the random disturbance U(t). Let us represent these parameters in the form

$$a_i = m_{a_i} + a_i^0 \ (i = 1, \ldots, \kappa);$$
$$b_j = m_{b_j} + b_j^0 \ (j = 1, \ldots, l),$$

where $m_{a_i}, m_{b_j}$ - nominal values of the parameters or their mathematical expectations;

$a_i^0, b_j^0$ - central random variables with preset dispersions $D_{a_i}, D_{b_j}$.

The nonlinearity of the type of the product of random coefficients to the i-th derivative or the variable enter into equation (II.126). Applying to these nonlinearity the methodology, presented in p. 1 of this chapter we will obtain

$$\left.\begin{array}{c} a_i Y^{(i)} = m_{a_i} m_{Y^{(i)}} + D_{a_i Y^{(i)}} + m_{a_i} Y^{0(i)} + m_{Y^{(i)}} a_i^0; \\ b_j U^{(j)} = m_{b_j} m_{U^{(j)}} + m_{b_j} U^{0(j)} + \\ + m_{U^{(j)}} b_j^0, \; i = 0, 1, \ldots, \kappa, \; j = 0, \ldots, l, \end{array}\right\} \quad \text{(II.127)}$$

where $D_{a_i Y^{(i)}}$ - moment/torque of the connection/communication of random coefficient $a_i$ and i derivative $Y^{(i)}$.

After the fulfillment of linearization we obtain system of equations for the mathematical expectations and random components of the variables:

$$E(s, m_{a_i}) m_Y + \sum_{r=0}^{\kappa} D_{a_r Y^{(r)}} = Q(s, m_{b_j}) m_U; \qquad \text{(II.128)}$$

$$E(s, m_{a_i}) Y^0 + Q(S, m_{b_j}) U^0 + + Q(s, b_j^0) m_U - E(s, a_i^0) m_Y. \qquad \text{(II.129)}$$

The system and disturbances/perturbations are stationary in the case in question.

Page 54.

We analyze the steady-state mode/conditions. In the steady-state mode/conditions from equation (II.128) we obtain

$$m_Y = \frac{m_{b_l}}{m_{a_\kappa}} m_U - \frac{D_{a_\kappa Y}}{m_{a_\kappa}}. \qquad \text{(II.130)}$$

Using equation (II.129), we compute values $D_Y$ and $D_{a_\kappa Y}$:

$$D_Y = \int_{-\infty}^{\infty} |W(j\omega, m_{a_l}, m_{b_l})|^2 S_U(\omega)\, d\omega +$$

$$+ \frac{m_U^2}{m_{a_\kappa}^2} D_{b_l} + \frac{m_Y^2}{m_{a_\kappa}} D_{a_\kappa}, \qquad \text{(II.131)}$$

where

$$W(j\omega, m_{a_l}, m_{b_l}) = \frac{Q(j\omega, m_{b_l})}{E(j\omega, m_{a_l})};$$

$$D_{a_\kappa Y} = -\frac{m_Y}{m_{a_\kappa}} D_{a_\kappa}. \qquad \text{(II.132)}$$

Obtained formulas (II.130) (II.131) (II.132) determine the mathematical expectations and dispersions of output variable taking into account the straggling of the parameters of systems.

The methodology presented easily is generalized to the system with the substantially nonlinear elements/cells with random parameters [31].

7. Combined statistical and harmonic linearization.

During the analysis of nonlinear systems, which are located in the self-vibrating mode/conditions or subjected to the interaction of periodic signal, in the presence of random disturbance can be used the method of statistical linearization in combination with the harmonic linearization. It is virtually important to consider the case, when at the input of nonlinear element/cell with characteristic F the signal

$$U = x_m \sin \omega_0 t + X,$$

where X(t) - random noise with mathematical expectation $m_x(t)$ and central random component X°(t), operates.

Page 55.

If F in the general case ambiguous nonlinearity, for example the type of relay with hysteresis, the output signal Y takes the form

$$Y = F(X + v, \dot{X} + \dot{v}), \qquad \text{(II.133)}$$

where

$$v(t) = x_m \sin \omega t.$$

For linearization of nonlinear dependence (II.133) let us use combined statistical and harmonic linearization. This can be done in two stages. It is preliminarily realized the statistical

linearization of function (II.133) relative to the random signal X(t), as it is recommended in p. 2 of this chapter. As a result we will obtain the linearized dependence, in which the statistical characteristic $F_0$ and statistical factors of amplification $k_1$ and $k_2$ will be the nonrandom periodic functions of the time:

$$\begin{aligned}Y(t) = F_0(m_X + x_m \sin\omega_0 t,\ \dot{m}_X + \omega x_m \cos\omega_0 t,\ D_X,\ D_{\dot{X}},\ D_{X\dot{X}}) +\\ + k_1(m_X + x_m \sin\omega_0 t, \dot{m}_X + \omega_0 x_m \cos\omega_0 t,\\ D_X,\ D_{\dot{X}},\ D_{X\dot{X}})X^0 + k_2(m_X + x_m \sin\omega_0 t,\ \dot{m}_X + \omega_0 x_m \cos\omega_0 t,\ D_X,\\ D_{\dot{X}},\ D_{X\dot{X}})\dot{X}^0.\end{aligned} \quad \text{(II.134)}$$

As a rule, mathematical expectations $m_X$ and $\dot{m}_X$, and also dispersions $D_X$, $D_{\dot{X}}$ are changed sufficiently slowly, and it is possible to consider as the their constants within the limits of the period of the harmonic component of signal. Then, realizing an harmonic linearization of functions $F_0$, $k_1$, $k_2$, let us expand them in Fourier series and let us preserve the necessary number of terms so as to exclude nonlinear components. As a result we will obtain

$$\begin{aligned}Y(t) = F_0^* + a^* x_m \cos\omega_0 t + b^* \omega_0 x_m \times\\ \times \sin\omega_0 t + k_1^* X^0 + k_2^* \dot{X}^0,\end{aligned} \quad \text{(II.135)}$$

where $F_0^*$ - statistical characteristic of nonlinearity averaged during the period of a change in the harmonic signal;

$k_1^*$ and $k_2^*$ - time-averaged statistical coefficients of the amplification of random components;

a*, b* - statistically averaged harmonic coefficients of the linearization of the statistical characteristic $F_0$.

Page 56.

These coefficients and averaged statistical characteristic are expressed by the formulas

$$\left.\begin{aligned}
F_0^* &= \frac{1}{2\pi}\int_0^{2\pi} F_0(m_X + x_m \sin\psi,\ x_m\cos\psi,\ D_X,\ D_{\dot X},\ D_{X\dot X})\,d\psi;\\
a^* &= \frac{1}{\pi x_m}\int_0^{2\pi} F_0(m_X + x_m \sin\psi,\ x_m\cos\psi,\ D_X,\ D_{\dot X},\ D_{X\dot X})\times\\
&\quad\times\cos\psi\,d\psi;\\
b^* &= -\frac{1}{\omega_0\pi x_m}\int_0^{2\pi} F_0(m_X + x_m \sin\psi,\ x_m\cos\psi,\ D_X,\ D_{\dot X},\\
&\quad D_{X\dot X})\sin\psi\,d\psi;\\
k_1^* &= \frac{1}{2\pi}\int_0^{2\pi} k_1(m_X + x_m \sin\psi,\ x_m\cos\psi,\ D_X,\ D_{\dot X},\ D_{X\dot X})\,d\psi;\\
k_2^* &= \frac{1}{2\pi}\int_0^{2\pi} k_2(m_X + x_m \sin\psi,\ x_m\cos\psi,\ D_X,\ D_{\dot X},\ D_{X\dot X})\,d\psi.
\end{aligned}\right\}\quad (\text{II}.136)$$

Functions $F_0$, $k_1$, $k_2$ in accordance with the method of statistical linearization are expressed by the formulas

$$\left.\begin{aligned}
F_0 &= \iint_{-\infty}^{\infty} F(x + v,\ \dot x + \dot v)\,p(x,\ \dot x)\,dx\,d\dot x;\\
k_1 &= \frac{\partial F_0}{\partial m_X};\quad k_2 = \frac{\partial F_0}{\partial m_{\dot X}}.
\end{aligned}\right\}\quad (\text{II}.137)$$

During the theoretical studies input signal is conveniently represented in complex form $U = X_m e^{j\omega_0 t} + X$. In this case output variable

also is represented in the complex form

$$Y = F_0^* + J^* x_m e^{j\omega_0 t} + k_1^* X^0 + k_2^* \dot{X}^0, \qquad (II.138)$$

where

$$J^* = a^* + jb^*.$$

Combined statistical harmonic linearization can be also used to the multidimensional nonlinearity of more general view [31].

8. Calculation of oscillations/vibrations in the nonlinear systems during the random disturbances.

Auto-oscillations are one of the possible modes/conditions of the work of nonlinear fixed systems. During the random disturbances the characteristics of the self-vibrating mode/conditions of system are changed up to the full/total/complete disruption/separation at the specific intensity of disturbances/perturbations.

Page 57.

The task of research of the dynamics of hunting systems in the presence of random disturbances consists of the study of the conditions for existence of auto-oscillations and of the determination of their amplitude and frequency. The approximation method of combined statistical and harmonic linearization is used for

the solution of this problem. The set-forth below procedure of calculation of auto-oscillations in the nonlinear system in the presence of random disturbances in essence is analogous to known to procedure harmonic linearization. Let us illustrate this methodology based on the example of system with one nonlinearity, whose dynamics is characterized by the equation

$$E(s)X + H(s)F(X, \dot{X}) = R(s)U, \qquad \text{(II.139)}$$

where F - nonlinearity, for example the type of relay with hysteresis;

U(t) - random stationary disturbance;

E(s), H(s), R(s) - polynomials relative to s with the constant coefficients.

Let us assume that the single-frequency auto-oscillations are possible in the system and system possesses sufficient filtering properties. Then in the steady-state mode/conditions we approximately record the variable X(t) in the form

$$X(t) = m_x + x_m e^{i\omega_0 t} + X^0(t), \qquad \text{(II.140)}$$

where $m_x = \text{const}$;

$x_m$, $\omega_0$ - amplitude and the frequency of fundamental harmonic component of auto-oscillations; $X^0(t)$ - central random component.

Let us replace nonlinearity with statistically equivalent linear dependence according to formula (II.138) under the assumption of a good filtration of oscillations:

$$F(X, \dot{X}) \quad k_0^* m_x + J^* x_m e^{j\omega_0 t} + k_1^* X^0 + k_2^* \dot{X}^0, \qquad (II.141)$$

where $k_0^* m_x \quad F_0^*$, and functions $F_0^*, k_1^*, k_2^*, J^*$ are determined by formulas (II.136). Substituting expressions (II.140) and (II.141) into equation (II.139) and separating/liberating constant oscillatory harmonic random component for the steady-state mode/conditions, we will obtain

$$E(0)\, m_X + H(0)\, k_0 m_X - R(0)\, m_U; \qquad (II.142)$$

$$E(s)\, \dot{x}_m e^{j\omega_0 t} + H(s)\, x_m J^* e^{j\omega_0 t} = 0; \qquad (II.143)$$

$$E(s)\, X^0 + H(s)\,(k_1^* + k_2^* s)\, X^0 = R(s)\, U^0. \qquad (II.144)$$

Equation (II.142) serves for determining the mathematical expectation $m_x$.

Page 58.

Equation (II.143) must be used for determining of amplitude $x_m$ and frequency of possible auto-oscillations $\omega_0$. For this purpose for the steady-state mode/conditions it is possible to convert it to the form

$$[E(j\omega_0)\, \dot{x}_m + H(j\omega_0)\, x_m J^*]\, e^{j\omega_0 t} = 0. \qquad (II.145)$$

The left side of equation (II.145) becomes zero, if expression in the brackets is equal to zero, i.e., characteristic equation has pure imaginary root:

$$E(j\omega_0) + H(j\omega_0)\,x_m J^* = 0. \qquad \text{(II.146)}$$

The left side of expression (II.146) is the complex quantity, for which it is possible to record [75]

$$\left.\begin{aligned} &\mathrm{Re}\,[E(j\omega_0)\,x_m + H(j\omega_0)\,x_m J^*] = 0,\\ &Jm\,[E(j\omega_0)\,x_m + H(j\omega_0)\,x_m J^*] = 0.\end{aligned}\right\} \qquad \text{(II.147)}$$

From these equations they determine $x_m$ and $\omega_0$ of auto-oscillations.

Equation (II.146) can be represented also in another form

$$W(j\omega_0) = -\frac{1}{J^*}. \qquad \text{(II.148)}$$

where

$$W(j\omega_0) = \frac{H(j\omega_0)}{E(j\omega_0)}.$$

From equation (II.144) let us determine $D_X, D_{\dot{X}}, D_{X\dot{X}}$:

$$D_X = \int_{-\infty}^{\infty} \left| \frac{R(j\omega)}{E(j\omega) + H(j\omega)(k_1^* + k_2^* j\omega)} \right|^2 S_U(\omega)\,d\omega; \qquad \text{(II.149)}$$

$$D_{\dot{X}} = \int_{-\infty}^{\infty} \left| \frac{R(j\omega)\,j\omega}{E(j\omega) + H(j\omega)(k_1^* + k_2^* j\omega)} \right|^2 S_U(\omega)\,d\omega; \qquad \text{(II.150)}$$

$$D_{X\dot{X}} = -\int_{-\infty}^{\infty} \left| \frac{R(j\omega)}{E(j\omega) + H(j\omega)(k_1^* + k_2^* j\omega)} \right|^2 j\omega S_U(\omega)\,d\omega, \qquad \text{(II.151)}$$

where $S_U(\omega)$ - the spectral density of random process U(t).

Equations (II.142) (II.147) (II.149) (II.150) (II.151) are actually equations for determination $m_x$, $x_m$, $D_x$, $D_{\dot{x}}$, $D_{x\dot{x}}$, $\omega_0$. They are connected with variables $k_0^*$, $k_1^*$, $J^*$, $k_2^*$, which depend on the same unknown values.

Page 59.

Connecting to these equations of the dependence

$$\left.\begin{aligned} k_0^* &= k_0^*(m_x x_m, D_x, D_{\dot{x}}, D_{x\dot{x}}),\\ k_1^* &= k_1^*(m_x, x_m, D_x, D_{\dot{x}}, D_{x\dot{x}}),\\ k_2^* &= k_2^*(m_x, x_m, D_x, D_{\dot{x}}, D_{x\dot{x}}),\\ J^* &= J^*(m_x, x_m, D_x, D_{\dot{x}}, D_{x\dot{x}}),\end{aligned}\right| \qquad (II.152)$$

we obtain the full/total/complete system, whose decision must be sought by successive approximations.

The method of successive approximations in this case can be applied, for example, in the form of the following algorithm.

We are assigned in the zero approximation by values $k_0^*$, $k_1^*$, $k_2^*$ and we determine from formulas (II.142) (II.149) (II.150) and (II.151) values $m_x$, $D_x$, $D_{\dot{x}}$, $D_{x\dot{x}}$ in the first approximation. This gives possibility on the basis of formula for J* to construct on the complex plane (Fig. II.12) the reverse amplitude-phase characteristic

of nonlinearity with the opposite sign $-\frac{1}{J^{*}(x_m)}$. The amplitude-phase characteristic of linear part $W(j\omega)=\frac{H(j\omega)}{E(j\omega)}$ is constructed in the same figure The point of their intersection, if it exists, in accordance with equation (II.145) determines values $x_m$ and $\omega_0$ in the first approximation. After using values $x_m, m_x, D_x, D_{\dot{x}}, D_{x\dot{x}}$, calculated in the first approximation, according to formulas (II.152) we compute new values $k_0^*, k_1^*, k_2^*$ and we repeat computations in the second approximation/approach. The procedure of calculations is finished in obtaining of the close consecutive values of the unknown values. The stability of auto-oscillatory mode/conditions is determined by ordinary methods [75] existing for this.

The random high-frequency disturbance, which operates on hunting system, changes the characteristics of auto-oscillations and, after achieving the specific level, it can tear away the self-oscillating process of system. If during the solution of the equations, in particular equations (II.147) given above or (II.148), it is not possible to determine actual values $x_m$ and $\omega_0$ (curves $W(j\omega)$ and $-1/J^*$ they do not intersect), then there is no self-oscillating process in the system.

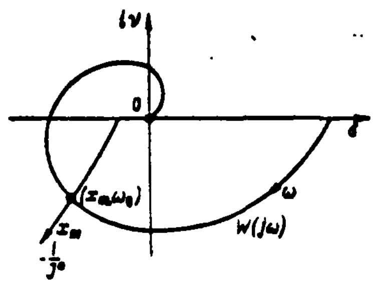



Fig. II.12. Amplitude-phase characteristic of linear part and nonlinearity.

Page 60.

The obtained formulas give the possibility to establish some limiting values of noise level, with which the auto-oscillations in the system cease. Thus, assuming/setting in these formulas $x_m = 0$, we will obtain equations for determining the critical values $m_x$ and $D_x$, at which in the system auto-oscillations cease.

Example. On the self-vibrating fixed system, which contains one nonlinear element/cell of the type of two-position relay with hysteresis, stationary white noise operates. The equations of system take the form

$$(c_0 s^3 + c_1 s^2 + c_2 s + c_3) Y = b_0 [F(x, s) + n];$$
$$X = U - Y, \qquad \text{(II.153)}$$

where n(t) - the stationary normally distributed white noise, which has spectral density $S_0$;

$m_n = 0, c_0, c_1, c_2, c_3, b_0$ - constant coefficients;

$$m_U = \text{const}.$$

The structural scheme of system is depicted in Fig. II.13. To determine the conditions for existence and stopping the natural oscillations.

For solving stated problem let us use the method of the statistical linearization of the ambiguous nonlinearity $F(X, \dot{X})$ on the assumption that single-valued auto-oscillations are possible in the system in the steady-state mode/conditions. Let us assume that in this case the variable X takes the form

$$X = m_X + X^0 + x_m \cos \omega_0 t. \qquad \text{(II.154)}$$

After the statistical linearization of function $F(X, \dot{X})$ on the basis of formula (II.141) we will obtain

$$F(X, \dot{X}) = F_0^* + k_1^* X^0 + k_2^* \dot{X}^0 + a^* x_m \cos \omega_0 t + + b^* x_m \omega_0 \sin \omega_0 t. \qquad \text{(II.155)}$$

where $F_0^*, k_1^*, k_2^*, a^*, b^*$ are determined from formulas (II.136) (II.137).

Substituting expression (II.155) into equation (II.153) and selecting constant component, periodic signal and central random

function, we will obtain system of equations for determining all components in the steady-state mode/conditions:

$$\left.\begin{array}{c} c_3 m_Y = b_0 F_0^*(m_X,\ m_{\dot{X}},\ D_X,\ D_{\dot{X}},\ D_{X\dot{X}},\ x_m); \\ m_X = m_U - m_Y; \end{array}\right\} \qquad (II.156)$$

$$\left(c_0 s^3 + c_1 s^2 + c_2 s + c_3\right) Y^0 = b_0 \left(k_1^* + k_2^* s\right) X^0 + n^0,\ X^0 = -Y^0. \qquad (II.157)$$

$$[c_0 (i\omega_0)^3 + c_1 (i\omega_0)^2 + c_2 (i\omega_0) + c_3] - b_0 (a^* + b^* i\omega_0) = 0. \qquad (II.158)$$

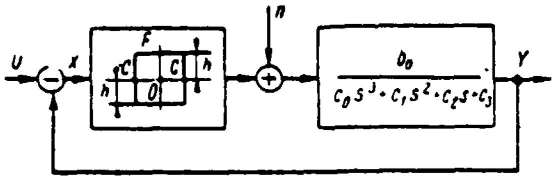



Fig. II.13. Nonlinear auto-oscillatory system.

Page 61.

Equation (II.158) let us represent in the form of expressions (II.147):

$$\left.\begin{aligned} -c_1\omega_0^2 + c_3 + b_0 a^* = 0; \\ -c_0\omega_0^3 + (c_2 + b_0 b^*)\,\omega_0 = 0. \end{aligned}\right\} \qquad \text{(II.159)}$$

From the second equation of system (II.159) we obtain the frequency of auto-oscillations

$$\omega_0 = \sqrt{\frac{c_2 + b_0 b^*}{c_0}}. \qquad \text{(II.160)}$$

First equation (II.159) taking into account formula (II.160) takes the form

$$-c_1(c_2 + b_0 b^*) + c_0 c_3 + b_0 a^* c_0 = 0. \qquad \text{(II.161)}$$

After using equation (II.157) and known formulas for computing the steady dispersions $D_x, D_X$ of stable stationary linear system, we will obtain

$$\left.\begin{aligned} D_X &= \frac{\pi b_0^2 S_0 c_1}{(c_3 + k_1^* b_0)\left[c_1(c_2 + k_2^* b_0) - c_0(c_3 + k_1^* b_0)\right]}; \\ D_{\dot{X}} &= \frac{c_3 + k_1^* b_0}{c_1} D_X;\ D_{X\dot{X}} = 0. \end{aligned}\right\} \quad \text{(II.162)}$$

Dependences (II.156) (II.160) (II.161) (II.162) together with the formulas for $F_0^*, k_1^*, k_2^*, a^*, b^*$ serve as the equations, which determine values $x_m, \omega_0, m_x, D_x, D_{\dot{x}}$. Their solution can be obtained by the method of successive approximations.

We analyze the obtained, for the formula in question, system for the purpose of the establishment of asymptotic relations. In the steady-state mode/conditions X(t) - stationary normal process. Therefore X(t) and $\dot{X}$(t) are not connected, while $m_x = 0$. In this case, formulas $F_0$, $k_1$ and $k_2$ for the relay in question take the form

$$\left.\begin{aligned}
F_0 = h\Big\{&\Phi\left(\frac{C+m_X+x_m\cos\omega_0 t}{\sqrt{D_X}}\right)-\Phi\left(\frac{C-m_X-x_m\cos\omega_0 t}{\sqrt{D_X}}\right)-2\Phi\times\\
&\times\left(\frac{x_m\omega_0\sin\omega_0 t}{\sqrt{D_{\dot X}}}\right)\Big[\Phi\left(\frac{C+m_X+x_m\cos\omega_0 t}{\sqrt{D_X}}\right)+\\
&+\Phi\left(\frac{C-m_X-x_m\cos\omega_0 t}{\sqrt{D_X}}\right)\Big]\Big\};\\
k_1 = \frac{h}{\sqrt{2\pi D_X}}\Big\{&\left[1+2\Phi\left(\frac{x_m\omega_0\sin\omega_0 t}{\sqrt{D_{\dot X}}}\right)\right]e^{-\frac{(C-m_X-x_m\cos\omega_0 t)^2}{2D_X}}+\\
&+\left[1-2\Phi\left(\frac{x_m\omega_0\sin\omega_0 t}{\sqrt{D_{\dot X}}}\right)\right]e^{-\frac{C+m_X+x_m\cos\omega_0 t}{2D_X}}\Big\};\\
k_2 = -\frac{2h}{\sqrt{2\pi D_{\dot X}}}\,&e^{-\frac{x_m^2\omega_0^2\sin^2\omega_0 t}{2D_{\dot X}}}\left[\Phi\left(\frac{C-m_X-x_m\cos\omega_0 t}{\sqrt{D_X}}\right)+\right.\\
&\left.+\Phi\left(\frac{C+m_X+x_m\cos\omega_0 t}{\sqrt{D_X}}\right)\right].
\end{aligned}\right\}\quad (\text{II.163})$$

Page 62.

Let $m_U = 0$. Then from expression (II.156), first formula (II.163) and first formula (II.136) it follows that $m_Y = 0,\ m_X = 0$. In this case from second and third formulas (II.136) in the extreme case (absence of auto-oscillations when $x_m \to 0$) we will obtain the limiting values of a* and b*:

$$a^* = \frac{2h\sqrt{2\pi}}{\sqrt{D_X}}\,e^{-\frac{C^2}{2D_X}};\quad b^* = -\frac{4h\sqrt{2\pi}}{\sqrt{D_{\dot X}}}\,\Phi\left(\frac{C}{\sqrt{D_X}}\right). \qquad (\text{II.164})$$

From formulas (II.163) it is possible to obtain also the values of coefficients $k_1^*$ and $k_2^*$ when $x_m = 0$:

$$k_1^* = \frac{2h}{\sqrt{2\pi D_X}}\,e^{-\frac{C^2}{2D_X}};\quad k_2^* = -\frac{4h}{\sqrt{2\pi D_{\dot X}}}\,\Phi\left(\frac{C}{\sqrt{D_X}}\right). \qquad (\text{II.165})$$

Substituting the limiting values of a*, b* in equation (II.161) and taking into account second formula (II.162) for $D_X$ and first formula (II.165) for $k_1^*$ we obtain the equation for determining the limiting value $D_X$, for which break loose themselves the auto-oscillations:

$$\left[c_0c_3 - c_1c_2 + c_0b_0 2h\sqrt{\frac{2\pi}{D_X}}\, e^{-\frac{C^2}{2D_X}}\right]\left(c_3 + \frac{2h}{\sqrt{2\pi D_X}}\, e^{-\frac{C^2}{2D_X}}\right) + + c_1b_0 4h\sqrt{2\pi}\,\Phi\left(\frac{C}{\sqrt{D_X}}\right) = 0. \qquad (II.166)$$

After the determination of limiting value $D_X$ from formulas (II.162) and (II.165) we find the limiting value of spectral density $S_0$ of the random disturbance n(t), during which break loose themselves the auto-oscillations:

$$S_0 > \frac{(c_3 + b_0k_1^*)\left[c_1(c_2 + b_0k_2^*) - c_0(c_3 + b_0k_1^*)\right]}{\pi b_0^2 c_1} D_X. \qquad (II.167)$$

Practical use in the control systems find nonlinear meters - filters for measuring the slowly changing signals in the presence of random steady noise. For an increase in the effective value of signal-to-noise ratio at the output of such nonlinear filters it is expedient to have the forced harmonic oscillatory or self-oscillating process with the acceptable amplitude of oscillations in the circuit of filter. Fig. II.14 depicts the standard structure of the nonlinear filter, whose equations take the form

$$Y = W(s)F(X);\; X = U - Y, \qquad (II.168)$$

where F(X) - nonlinearity of the type of clipping or ideal relay;

W(s) - rational-linear operator.

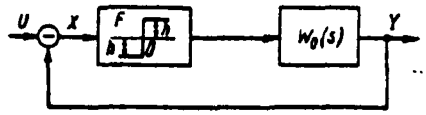



Fig. II.14. Nonlinear filter.

Page 63.

Let the input signal U contain slowly varying component $U_0$, which it is necessary in the best way to isolate, harmonic fluctuating component $a_0 \sin\omega_0 t$ and the stationary random noise $U^\circ(t)$, which has equal to zero mathematical expectation and spectral density

$$S_U(\omega) = D_0 \frac{\beta}{\pi} \cdot \frac{1}{\omega^2 + \beta^2}.$$

Let us consider different cases of filtrating the slowly varying signal $U_0$ by this filter. In this case we will consider signal $U_0$ as that slowly changing or in effect constant, if its change can be disregarded/neglected during the period of oscillatory process $2\pi/\omega_0$.

Case 1. Let $W(s) = \frac{k}{Ts+1}$; F(X)=sign X; $a_0=0$, i.e., oscillational component is absent. Signal-to-noise ratio at the input into the filter

$$q_0 = \frac{U_0}{\sqrt{D_0}}.$$

Let us determine signal-to-noise ratio at the output of the filter:

$$q_Y = \frac{m_Y}{\sqrt{D_Y}}.$$

For this it is realized the statistical linearization of nonlinearity, also, from the obtained linear equations for the steady-state mode/conditions we find

$$\left.\begin{aligned} m_Y &= \frac{kk_0(m_X,\ D_X)}{1+kk_0(m_X,\ D_X)}U_0; \\ D_Y &= \frac{D_0k^2k_1^2(m_X,\ D_X)}{[1+kk_1(m_X,\ D_X)][T\beta+1+kk_1(m_X,\ D_X)]}; \end{aligned}\right\} \quad (II.169)$$

$$\left.\begin{aligned} m_X &= \frac{U_0}{1+kk_0\ (m_X,\ D_X)}; \\ D_X &= \frac{D_0(T+1)}{[1+kk_1(m_X,\ D_X)][T\beta+1+kk_1(m_X,\ D_X)]}; \end{aligned}\right\} \quad (II.170)$$

$$k_0 = \frac{2}{m_X}\Phi\left(\frac{m_X}{\sqrt{D_X}}\right);$$

$$k_1 = \frac{2}{\sqrt{2\pi D_X}}e^{-\frac{m_X^2}{2D_X}}.$$

Page 64.

Signal-to-noise ratio at the output of filter in this case takes the form

$$q_Y = q_0\frac{k_0\sqrt{(1+kk_1)(T\beta+1+kk_1)}}{k_1(1+kk_0)}. \quad (II.171)$$

Example. Let the filter and input signal have the following parameters: k=1; T=1; $D_0$=1; $\beta$=0.5 and the arbitrary, but constant

value $U_0$=0-4. We preliminarily make calculations according to formulas (II.170) for the given values of the parameters and we compute $m_x, D_x$ graphically for each value of $U_0$ employing the procedure, presented in p. 3. In this case we determine coefficients of $k_0$ and $k_1$. Further according to formula (II.171) we compute signal-to-noise ratio $q_Y$ at different values of $q_0$. Fig. II.15 gives the final graph/diagram of dependence $q_Y$ on $q_0$ for the examined case (curve 1). In the same place for comparison is given dependence (line 2) for the linear filter, when $k_0 = k_1 = 1$; $q_Y = q_0\sqrt{1+\frac{T\beta}{1+k}}$.

Case 2. Let W(s)=k/s; F(X)=sign X and fluctuating component be absent ($a_0$=0). After leading the statistical linearization of nonlinearity, from the linear equations in the steady-state mode/conditions we will obtain the formulas

$$m_Y = u_0;\; D_Y = D_0\frac{kk_1(D_X)}{\beta + kk_1(D_X)}; \qquad \text{(II.172)}$$

$$m_X = 0;\; D_X = D_0\frac{\beta}{\beta + kk_1(D_X)};\quad k_1 = \frac{2}{\sqrt{2\pi D_X}}. \qquad \text{(II.173)}$$

In this case the system of equations (II.173) can be solved relatively $\sigma_X = \sqrt{D_X}$:

$$\sigma_X = -\frac{k}{\beta\sqrt{2\pi}} + \sqrt{\frac{k^2}{2\pi\beta^2} + D_0}. \qquad \text{(II.174)}$$

Ratio of signal $m_Y$ to noise $\sqrt{D_Y}$ at the output of the filter

$$q_Y = q_0\sqrt{\frac{\beta + kk_1(D_X)}{kk_1(D_X)}}.$$

Substituting the value $k_1(D_X)$ for the present instance and expression for $D_X$ in relationship/ratio (II.174), we will obtain finally

$$q_Y = q_0\sqrt{1+\frac{1}{2}\sqrt{1+\frac{2\pi\beta^2 D_0}{k^2}}-\frac{1}{2}}. \qquad \text{(II.175)}$$

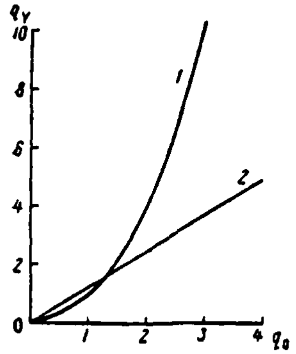



Fig. II.15. Signal-to-noise ratio is for the nonlinear filter.

Page 65.

If filter is linear with $k_1=1$, then signal-to-noise ratio at the output of filter is equal

$$q_Y = q_0\sqrt{1+\frac{\beta}{k}}.$$

Since

$$\frac{1}{2}\left(\sqrt{1+\frac{2\pi\beta^2 D_0}{k^2}}-1\right) > \frac{\beta}{k},$$

the signal-to-noise ratio at the output of nonlinear filter is greater than analogous relation at the output of linear filter.

Case 3. Let there be $W(s) = \frac{k}{s(Ts+1)}$; F(X)=sign X harmonic fluctuating component in the input signal, which for convenience in

the calculations let us take in the form $a_0e^{j\omega_0 t}$.

Assuming/setting in this case signal at the output of nonlinearity in the form

$$X = m_X + a_X e^{j(\omega_0 t + \alpha_X)} + X^0, \qquad \text{(II.176)}$$

where $\alpha_X$ – phase displacement of oscillation, which appears in the system, let us perform the combined statistical and harmonic linearization of the nonlinearity

$$\operatorname{sign} X = k_0^* m_X + a^* a_X e^{j(\omega_0 t + \alpha_X)} + k_1^* X^0, \qquad \text{(II.177)}$$

where $k_0^*, a^*, k_1^*$ - averaged statistical amplification factors for this nonlinearity, determined from formulas (II.136):

$$\left.\begin{aligned} k_0^* &= \frac{1}{m_X}\frac{1}{2\pi}\int_0^{2\pi} F_0(m_X + a_X\sin\psi, D_X)\,d\psi; \\ a^* &= \frac{1}{\pi a_X}\int_0^{2\pi} F_0(m_X + a_X\sin\psi, D_X)\sin\psi\,d\psi; \\ k_1^* &= \frac{1}{2\pi}\int_0^{2\pi} k_1(m_X + a_X\sin\psi, D_X)\,d\psi. \end{aligned}\right\} \qquad \text{(II.178)}$$

Functions $F_0$ and $k_1$ in this case take the form

$$F_0 = 2h\Phi\left(\frac{m_X + a_X\sin\psi}{\sqrt{D_X}}\right);\ k_1 = \frac{2h}{\sqrt{2\pi D_X}}e^{-\frac{(m_X + a_X\sin\psi)^2}{2D_X}}. \qquad \text{(II.179)}$$

Page 66.

Substituting expression (II.177) in equations (II.168), and also taking into account that the first harmonic oscillatory component of variable Y has form $a_Y e^{i(\omega_0 t + \alpha_Y)}$, we will obtain in the steady-state mode/conditions

$$\left.\begin{aligned} m_X &= 0;\\ m_Y &= u_0;\\ a_Y e^{j\alpha_Y} &= \frac{ka^* a_X}{T(j\omega_0)^2 + j\omega_0} e^{j\alpha_X};\\ a_X e^{j\alpha_X} &= a_0 - c_Y e^{j\alpha_Y};\\ (Ts^2 + s)\,Y^0 &= kk_1^* X^0;\\ X^0 &= U^0 - Y^0. \end{aligned}\right\} \qquad (II.180)$$

From the obtained linear equations it is possible to obtain formulas for determining of amplitude $a_X$ and dispersion $D_X$:

$$a_0^2 = a_X^2 \frac{(ka^* - T\omega_0^2)^2 + \omega_0^2}{T^2\omega_0^4 + \omega_0^2}; \qquad (II.181)$$

$$D_X = D_0\beta \frac{1 + (\beta + k_1^* k)\,T}{(T\beta + 1)\,\beta + kk_1^*}, \qquad (II.182)$$

where coefficients $k_1^*(a_X, D_X)$ and $a^*(a_X, D_X)$ for the ideal relay are calculated from formulas (II.178) when $m_X = 0$ and they take form [69]

$$a^*(a_X,\ D_X) = \frac{1}{a_X} B_0(\alpha);\quad k_1^*(a_X,\ D_X) = \frac{1}{\sqrt{D_X}} C_0(\alpha), \qquad (II.183)$$

where

$$B_0(\alpha) = \frac{4}{\sqrt{\pi}} \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa+n}[2(\kappa+1)]!}{(2n)!\,(\kappa!)^2\,(\kappa+n)!\,(\kappa+1)} \left(\frac{\alpha}{2}\right)^{2\kappa+1};$$

$$C_0(\alpha) = \sqrt{\frac{2}{\pi}} \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa+n}[2(k+n)]!}{(2n)!\,(\kappa!)^2\,(\kappa+n)!} \left(\frac{\alpha}{2}\right)^{2\kappa}, \quad n = 0, 1, 2, \ldots$$

$$\alpha = \frac{a_X}{\sqrt{2D_X}}.$$

Solving equations (II.181) (II.182) upon consideration of formulas (II.183), let us determine values $a_X$ and $D_X$ and, therefore, value $a^*, k_1^*$.

Page 67.

Further from equations (II.180) for determining the amplitude $a_Y$ of the fundamental harmonic of oscillations at the output of system and dispersion $D_Y$ we obtain the formulas

$$a_Y = \frac{ka^* a_0}{\sqrt{(ka^* - T^2\omega_0^2)^2 + \omega_0^2}}; \qquad \text{(II.184)}$$

$$D_Y = D_0 kk_1^* \frac{1 + T\beta}{kk_1^* + T\beta^2 + \beta}. \qquad \text{(II.185)}$$

Signal-to-noise ratio at the input of filter is equal

$$q_0 = \frac{u_0}{\sqrt{D_0 + \frac{1}{2} a_0^2}},$$

while at the output of nonlinear filter in this case this relation is calculated from the formula

$$q_Y = \frac{u_0}{\sqrt{D_Y + \frac{1}{2} a_Y^2}}.$$

Page 68.

## Chapter III.

## ANALYSIS OF RANDOM PROCESSES IN NONLINEAR SELF-TUNING SYSTEMS.

### 1. Conditions for work of nonlinear self-tuning systems.

The fundamental information about the nonlinear self-tuning systems, and also the methods of the study of auto-oscillations and transient processes in these systems is presented in the book "Method of harmonic linearization in the design of the nonlinear automatic control systems" M., "Machine building", 1970. Therefore here we will be bounded only to the presentation of the methodology of the study of some nonlinear self-tuning systems under the influence of random signals. It should be noted that in the general case the fundamental outline of the self-tuning system is subjected to the interaction of two types of the disturbances/perturbations: parametric ones and signal ones. Parametric disturbances/perturbations are determined by changes in the parameters of the object of control; in certain cases the changes in the parameters of regulator, caused, for example, by

aging or breakdown of the single elements/cells of regulator, are also the source of parametric disturbances/perturbations.

Page 69.

In the general case parametric disturbances/perturbations can be described by certain random function of time. Frequently the parametric disturbances/perturbations are multiplicative interference with respect to the useful signal, passing through the object of control. Let us assume, for example, that in the object of control randomly is changed the factor of amplification K(t), and the harmonic signal A sin $\omega$t enters the input of object. Usually the band of the frequencies considered in the spectral plane of function K(t) is arranged/located much lower than frequency $\omega$ of harmonic input signal. In other words, function K(t) is the slowly varying random function (in comparison with the input signal). Then signal X(t) at the output of the object of control is approximately described by the formula

$$X(t) \quad K(t) A_1 \sin(\omega t + \varphi),$$

where $A_1$ and $\varphi$ are determined through the transfer function of the object of control.

Since parametric disturbances/perturbations are usually described by the slowly varying functions of time, then during the

research of the self-tuning systems with respect to the disturbances/perturbations indicated is applied the method of the "frozen coefficients". It should be noted that the parametric disturbances/perturbations in the fundamental outline are the source of useful (with respect to the self-tuning loop) signals, since the task of self-tuning loop consists of the evaluation/estimate of parametric disturbances on the basis of the analysis of the signals of fundamental outline and of the compensation for the parametric disturbances/perturbations by changing the parameters of regulator. Signal disturbances/perturbations operate on the automatic system together with the control pressure. Usually signal disturbances/perturbations are additive interference, i.e., signal S(t), which influences the fundamental outline, is determined by the formula

$$S(t) \quad u(t)+f_1(t),$$

where u(t) - the control pressure;

$f_1(t)$ - the interference (signal disturbance/perturbation), led to the input of the fundamental outline of system.

In certain cases of the interferences, which influence the fundamental outline, they are used for the self-adjusting. Let us consider, for example, the self-tuning system with two frequency filters [39], whose simplified block diagram is given in Fig. III.1.

The fundamental outline of the system consists of object O, on which acts the signal disturbance/perturbation f(t), the unit of the varied coefficient of regulator BK and amplifier Y. Self-tuning loop contains two channels - channel of an increase in the varied coefficient and channels of its decrease.

Page 70.

The first channel consists of high-frequency band-pass filter $\Phi_в$ and detector D, second channel - from low-frequency band-pass filter $\Phi_н$ and detector D. Signal the self-tuning loop enters from point a (error signal $\epsilon$ of fundamental outline) or from point b (output signal of the X(t) fundamental outline). Difference $z_н - z_в$ between the signals of respectively high-frequency and low-frequency channels enters integrator of the I self-tuning loop, and then the unit of the varied coefficient BK.

The principle of the work of the self-tuning system in question consists of the following.

If the fundamental outline of the self-tuning system approaches a stability limit, then the fraction/portion of high-frequency components in comparison with the low-frequency ones increases in the spectrum of the error signal $\epsilon$(t). Therefore difference $z_н - z_в$

becomes negative, which causes the decrease of the varied coefficient until the difference indicated becomes equal to zero. During the considerable removal/distance from the stability limit the fundamental outline of the system becomes slow-acting, and therefore increases the fraction/portion of low-frequency components in comparison with the high-frequency ones. Difference $z_{н} - z_{ч}$ becomes positive, and integrator И feeds the signal of an increase in the coefficient. At the stationary signal disturbances/perturbations the preset stability factor is automatically supported thus in the system, in spite of changes in the factor of amplification of object.

In certain cases in the self-tuning systems of interference they are capable of substantially worsening/impairing the work of system, and sometimes making it inefficient. Therefore the account of the interaction of interferences (signal disturbances/perturbations) is frequently the necessary stage during the design of the self-tuning system.

Let us consider, for example, the block diagram of the self-tuning system, given in Fig. III.2.

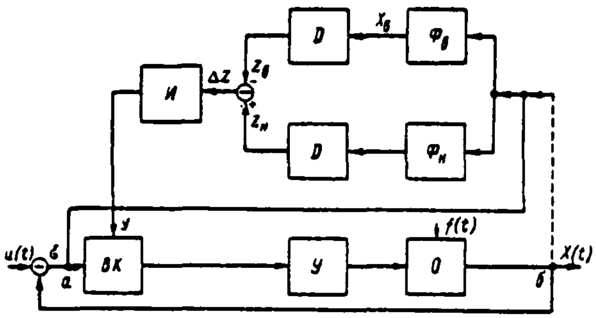



Fig. III.1. Self-tuning system with two frequency filters.

Page 71.

Fundamental outline contains the unit of controlled parameters БРП and the object of control O. Self-tuning loop consists of the filter Φ of the nonlinear converter НП, which determines the algorithm of self-adjusting, and the actuating element of self-tuning loop ИУ. The input of the fundamental outline of system enter control pressure u(t) and test signal g(t), developed by the generator of test signal ГПС. Interference f(t) is applied to the object of control O. Pulse or harmonic signals are commonly used as the test signals. Test signal is absent in certain cases, then the evaluation/estimate of the quality of the work of system is realized with the help of the analysis of the auto-oscillations of the fundamental outline of system[1].

FOOTNOTE [1]. See the "method of harmonic linearization in the design of nonlinear automatic control systems". ("Nonlinear automatic control systems"). M., "Machine building", 1970. ENDFOOTNOTE.

The given block diagram corresponds to different types of the self-tuning systems, examined in the literature [64].

It is known that with the effect of interferences on ordinary hunting system occurs a change in the amplitude of auto-oscillations, and upon reaching/achievement of the specific interference level auto-oscillations cease [34], [69]. The structural scheme of the self-vibrating self-tuning system is given in Fig. III.3.

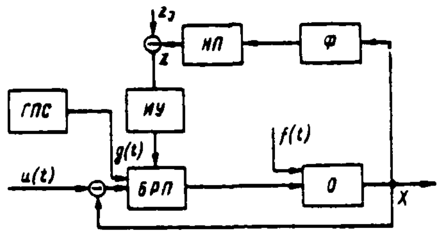



Fig. III.2. Block diagram of the self-tuning system.

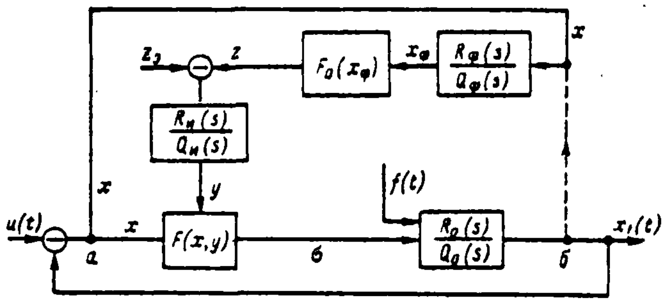



Fig. III.3. Structural scheme of the self-tuning system.

Page 72.

On this diagram through $R_0(s)/Q_0(s)$ is designated the transfer function of the object of control, and through $\frac{R_\Phi(s)}{Q_\Phi(s)}$ and $\frac{R_U(s)}{Q_U(s)}$ - with respect the transfer functions of filter and actuating element of self-tuning loop. Research of processes in similar systems is made frequently with the help of the method of statistical linearization. Therefore is natural the examination of the possibility of applying this method, also, to the nonlinear self-tuning systems, which are more complicated than ordinary nonlinear automatic systems.

The target of this chapter is the short presentation of the specific special features/peculiarities of the use/application of methods of statistical and harmonic linearization to the nonlinear self-tuning systems.

2. Statistical coefficients of the linearization of the standard nonlinearity of the self-tuning systems.

From the structural scheme of the search-free self-tuning system given in Fig. III.3 it is evident that the systems in question contain at least two nonlinearity: nonlinearity F(x, y), which characterizes the parametric connection/communication of fundamental outline with the self-tuning loop, and nonlinearity $F_0(x, \varphi)$, determined by the selected algorithm of self-tuning loop. Since the nonlinearity indicated differ somewhat from the standard nonlinearity of ordinary automatic systems, then is usefully to preliminarily consider the possible methods of the statistical linearization of the nonlinearity of the form indicated.

Statistical coefficients of the linearization of nonlinearity F(x, y). Nonlinearity F(x, y) is in the general case function two random dependent the variables X and Y. Let us consider the case,

when self-tuning loop is completely interference-free, at first, i.e., coordinate y is either constant value or value of that of slowly varying in the conformity only with a change in the parameters of fundamental outline. The position indicated occurs in such a case, when, for example, the spectrum of random noise lies/rests out of the filter pass band, which is located in the self-tuning loop. Then to nonlinearity F(x, y) it is possible to use the principle of the separate harmonic linearization[1], in accordance with which it is possible by ordinary methods to carry out the statistical linearization of nonlinearity F(x, y), by considering y as the parameter.

FOOTNOTE [1]. See the book "method of harmonic linearization in the design of nonlinear automatic control systems". ("Nonlinear automatic control systems"). M., "Machine building", 1970. ENDFOOTNOTE.

The case, when together with useful constant or slowly varying component value y contains certain random component of Y, is more complicated.

Page 73.

Applying the general formula of the linearization of multidimensional functional dependence [31], we will obtain

$$\rho = F(x, y) = \rho_0 + k_{\sigma_x} X + k_{\sigma_y} Y, \qquad \text{(III.1)}$$

where $\rho_0$ - statistical characteristic of nonlinear element/cell;

$k_{\sigma_x}$ and $k_{\sigma_y}$ - statistical coefficients of linearization on random components X and Y.

Let us determine the statistical coefficients of linearization from the condition

$$\mathrm{M}\{[F(x, y) - \rho_0 - k_{\sigma_x} X - k_{\sigma_y} Y]^2\} = \min. \qquad \text{(III.2)}$$

Equalizing to zero partial derivatives on $\rho_0$, $k_{\sigma_x}$ and $k_{\sigma_y}$ of the left side of relationship/ratio (III.2), we will obtain

$$\rho_0 = m_\rho = \mathrm{M}[F(x, y)]; \qquad \text{(III.3)}$$

$$k_{\sigma_x} = \frac{D_y R_{\rho x} - R_{\rho y} R_{xy}}{D_x D_y - R_{xy}^2}; \qquad \text{(III.4)}$$

$$k_{\sigma_y} = \frac{D_x R_{\rho y} - R_{\rho x} R_{xy}}{D_x D_y - R_{xy}^2}, \qquad \text{(III.5)}$$

where $m_\rho$ - mathematical expectation of variable $\rho$, determined by the formula

$$m_\rho = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x, y)\, p(x, y)\, dx\, dx; \qquad \text{(III.6)}$$

$D_x, D_y$ - the dispersion of the variables x and y;

$R_{xy} = R_{xy}(t, t)$, $R_{\rho x} = R_{\rho x}(t, t)$, $R_{\rho y} = R_{\rho y}(t, t)$ - the cross-correlation functions of the corresponding variables.

In formula (III.6) p(x, y) - the two-dimensional density of

array of the variables x and y. It is not difficult to show (see Chapter II, and also work [31] that under the normal law of array of the variables x and y the statistical coefficients of linearization $k_{\sigma_x}$ and $k_{\sigma_y}$ are connected with value $m_\rho$ with the following relationships/ratios;

$$k_{\sigma_x} = \frac{\partial m_\rho}{\partial m_x}; \quad k_{\sigma_y} = \frac{\partial m_\rho}{\partial m_y}. \qquad \text{(III.7)}$$

It is evident from formulas (III.4)-(III.6) that in the general case of the correlated random functions X and Y the expressions for values $m_\rho$, $k_{\sigma_x}$ and $k_{\sigma_y}$ are sufficiently complicated (especially, if we take into account the difficulties of determination explicitly $R_{xy}$, $R_{\rho x}$ and $R_{\rho y}$). This circumstance substantially impedes research of the nonlinear self-tuning systems.

Page 74.

Therefore is of interest the analysis of the conditions, under which the functions indicated can be considered as not correlated (but in the case of normal processes and by independent variables). Comprehensive research of these conditions is difficult; therefore we will be bounded to the approximate conclusions/outputs, based on the simplified approach to the analysis of the interconnection of these values, that considers some simple physical considerations.

It follows (See Fig. III.3) from the given earlier structural scheme of the nonlinear self-tuning system that in the general case signal y is obtained usually as the result of the consecutive use/application of linear, nonlinear and then again linear transformations of signal with the help of the components/links, which have respectively characteristics $W_{\Phi}(s)=\frac{R_{\Phi}(s)}{Q_{\Phi}(s)}$; $F_0(x_{\Phi})$ and $W_H=\frac{R_H(s)}{Q_H(s)}$. If we are bounded to the examination of low-frequency random processes, then in the transfer functions, which correspond to the linear transformations indicated, it is possible to assume s=0. Consequently, self-tuning loop with respect to slowly varying random component can be considered as the nonlinear inertia-free device/equipment, described by the function

$$y=k_2F_0(k_1X), \qquad \text{(III.8)}$$

where coefficients $k_1$ and $k_2$ are determined by the formulas

$$k_1=W_{\Phi}(0);\quad k_2=W_H(0).$$

Let us disconnect self-tuning loop at the output of actuating element and will consider the equivalent diagram, given in Fig. III.4.

Regarding, the cross-correlation function of variables $y=m_y+Y$ and $x=m_x+X$ is determined by the formula

$$R_{xy}(t_1,\ t_2)=\mathrm{M}[X(t_1)Y(t_2)].$$

With $t_1=t_2=t$ taking into account relationship/ratio (III.8) we have

$$R_{xy}(t,\ t) = k_2\mathrm{M}\,[x(t)\,F_0(k_1x)] = k_2\int\limits_{-\infty}^{\infty} xF_0(k_1x)\,p(x)\,dx\,.$$

Let us show that if function $F_0(k_1X)$ - is even symmetrical, then for normal random process $R_{xy}(t,\,t) = 0$.

Actually/really

$$R_{xy}(t,\ t) = k_2\left[\int\limits_{-\infty}^{0} xF_0(-k_1x)\,p(x)\,dx + \int\limits_{0}^{\infty} xF_0(k_1x)\,p(x)\,dx\right].$$

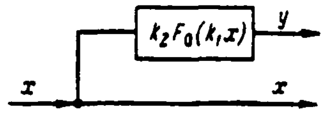



Fig. III.4. Simplified equivalent diagram of self-tuning loop.

Page 75.

Since

$$F_0(-k_1x) = F_0(k_1x) \text{ and } p(-x) = p(x),$$

then

$$\int_{-\infty}^{0} xF_0(-k_1x)\,p(-x)\,dx = -\int_{0}^{\infty} xF_0(k_1x)\,p(x)\,dx,$$

i.e.

$$R_{xy}(t,\ t) = 0.$$

The obtained result is strict (when making these assumptions) for the case of the extended self-tuning loop; during closing/shorting of self-tuning loop random component of Y by the variable y affects variable -x, and therefore the conclusion obtained above about the noncorrelation of the variables x and y for the locked self-tuning loop is, generally speaking, inaccurate. However, if random component Y is small (which frequently occurs), then the effect of random component Y indicated on the variable x is also small, and in the first approximation, it is possible to consider that and in this case $R_{xy}(t,\ t) \approx 0$. In accordance with formulas (III.4)

and (III.5) when $R_{xy}=0$ we have

$$k_{\sigma_x}=\frac{R_{\rho x}}{D_x};\quad k_{\sigma_y}=\frac{R_{\rho y}}{D_y}.$$

Thus, in the case in question statistical coefficient $k_{\sigma_x}$ of linearization on random component X proves to be independent from the statistical characteristics random component Y, and $k_{\sigma_y}$ - from the statistical characteristics of random component X.

Frequently the nonlinear function F(x, y) can be represented in the form of the product of two functions:

$$\rho=F(x,\ y)=\lambda(x)\nu(y). \qquad \text{(III.9)}$$

Then, taking into account that for uncorrelated the variables X and Y probability density p(x, y)=p(x)p(y), we will obtain:

$$m_\rho=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(x,\ y)\,p(x,\ y)\,dx\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\lambda(x)\nu(y)\,p(x)\,p(y)\,dx\,dy=$$
$$=\int_{-\infty}^{\infty}\lambda(x)\,p(x)\,dx\int_{-\infty}^{\infty}\nu(y)\,p(y)\,dy=m_\lambda m_\nu, \qquad \text{(III.10)}$$

where $m_\lambda$ and $m_\nu$ - respectively mathematical expectations of functions $\lambda(x)$ and $\nu(y)$.

Page 76.

Frequently $\nu(y)$ is linear function, i.e., $\nu(y)=ky$. Then from expression (III.10) we have

$$m_\rho \quad k m_y m_\lambda, \tag{III.11}$$

and, therefore, in accordance with formula (III.7) for $k_{\sigma_y}$ we have

$$k_{\sigma_y} = k m_\lambda. \tag{III.12}$$

From formula (III.12) follows that if function F(x, y), determined by formula (III.9), is odd symmetrical relative to x, i.e., $\lambda(x) = -\lambda(-x)$ and $m_x = 0$, then $k_{\sigma_y} = 0$. Let us consider as an example of the formulas, which determine $\rho_0$, $k_{\sigma_x}$ and $k_{\sigma_y}$ relay element/cell with the varied level of the limitation:

$$\rho = F(x, y) = y \operatorname{sign} x. \tag{III.13}$$

Substituting in formula (III.6)

$$p(x, y) = \frac{1}{2\pi\sqrt{D_x D_y - R_{xy}^2}} \exp \times$$
$$\times \left\{ -\frac{D_y (x - m_x)^2 - 2R_{xy}(x - m_x)(y - m_y) + D_x (y - m_y)^2}{2(D_x D_y - R_{xy}^2)} \right\},$$

we will obtain

$$\rho_0 = \frac{2m_y}{\sqrt{CD_x}} \Phi(m_x \sqrt{C}) + \frac{2R_{xy}}{D_x^{\frac{1}{2}} \sqrt{2\pi C}} e^{-\frac{Cm_x^2}{2}}, \tag{III.14}$$

where

$$C = \frac{1}{D_x D_y - R_{xy}^2} \left( D_y + \frac{R_{xy}^2}{D_x} \right);$$

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{t^2}{2}} dt.$$

In accordance with formulas (III.7) we will obtain

$$k_{\sigma_x} = \sqrt{\frac{2}{\pi}}\left(m_y - \frac{\sqrt{C}\,R_{xy}m_x}{\sigma_x^2}\right)\frac{\exp\left(-\frac{m_x^2 C}{2}\right)}{\sigma_x}; \quad \text{(III.15)}$$

$$k_{\sigma_y} = \frac{2}{\sigma_x\sqrt{C}}\,\Phi\left(m_x\sqrt{C}\right). \quad \text{(III.16)}$$

Page 77.

If $m_x = 0$, then

$$\rho_0 = \frac{2R_{xy}}{\sigma_x\sqrt{2\pi C}}; \quad k_{\sigma_x} = \sqrt{\frac{2}{\pi}}\cdot\frac{m_y}{\sigma_x}; \quad k_{\sigma_y} = 0. \quad \text{(III.17)}$$

If the variables X and Y are independent variables and, therefore, $R_{xy} = 0$, then from formulas (III.14)-(III.16) it follows

$$\rho_0 = 2m_y\Phi\left(\frac{m_x}{\sigma_x}\right); \quad \text{(III.18)}$$

$$k_{\sigma_x} = \sqrt{\frac{2}{\pi}}\frac{m_y}{\sigma_x}\exp\left(-\frac{m_x^2}{2\sigma_x^2}\right); \quad \text{(III.19)}$$

$$k_{\sigma_y} = 2\Phi\left(\frac{m_x}{\sigma_x}\right). \quad \text{(III.20)}$$

Let us consider the task of the statistical linearization of nonlinearity F(x, y), if one of the variables, for example x, besides random central component X contains also the harmonic component of A sin ωt. If the variable y is the constant or slowly varying value, then, by considering it as the parameter, the approximating function

let us take in the form

$$\rho = F(x, y) = \rho_0 + \varkappa A \sin \omega t + \varkappa_1 X, \qquad (III.21)$$

where $\varkappa(A, m_x, \sigma_x, y)$ - statistical coefficient of linearization on periodic (harmonic) component;

$\varkappa_1(A, m_x, \sigma_x, y)$ - statistical coefficient of linearization on random component during the nonlinear conversion of the sum of random normal and harmonic of signals.

Values $\rho_0$, $\varkappa$ and $\varkappa_1$ are determined by the formulas

$$\rho_0 = m_\rho = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} F(x + A \sin \varphi, y)\, p(x)\, dx; \qquad (III.22)$$

$$\varkappa = \frac{1}{\pi A} \int_0^{2\pi} \sin \varphi\, d\varphi \int_{-\infty}^{\infty} F(x + A \sin \varphi, y)\, p(x)\, dx; \qquad (III.23)$$

$$\varkappa_1 = \frac{1}{2\pi\sigma_x^2} \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} F(x + A \sin \varphi, y)\, x p(x)\, dx. \qquad (III.24)$$

Page 78.

In particular, for the relay element/cell with the varied level of limitation we have when $m_x = 0$ [69]

$$\rho_0 = 0; \quad \varkappa(A, \sigma_x, y) = \frac{y}{A} B_0(\alpha); \quad \varkappa_1(A, \sigma_x, y) = \frac{y}{\sigma_x} C_0(\alpha),$$

where

$$\alpha = \frac{A}{\sqrt{2}\sigma_x};$$

$$B_0(\alpha) = \frac{4}{\pi}\left[1 - \frac{1}{(2\alpha)^2} - \frac{3}{2}\cdot\frac{1}{(2\alpha)^4} - \frac{15}{2}\cdot\frac{1}{(2\alpha)^6} - \dots\right];$$

$$C_0(\alpha) = \frac{\sqrt{2}}{\pi\alpha}\left[1 + \frac{1}{(2\alpha)^2} + \frac{9}{2}\cdot\frac{1}{(2\alpha)^4} + \frac{75}{2}\cdot\frac{1}{(2\alpha)^6} + \dots\right]. \quad \text{(III.25)}$$

The given formulas are convenient at the high values $\alpha$; with small ones $\alpha$ they are applied other expressions for functions $B_0(\alpha)$ and $C_0(\alpha)$:

$$\left.\begin{aligned} B_0(\alpha) &= \frac{4}{\sqrt{\pi}}\sum_{\kappa=0}^{\infty}\frac{(-1)^{\kappa}(2\kappa)!}{(\kappa!)^3(\kappa+1)}\cdot\left(\frac{\alpha}{2}\right)^{2\kappa+1}; \\ C_0(\alpha) &= \sqrt{\frac{2}{\pi}}\sum_{\kappa=0}^{\infty}\frac{(-1)^{\kappa}(2\kappa)!}{(\kappa!)^3}\cdot\left(\frac{\alpha}{2}\right)^{2\kappa}. \end{aligned}\right\} \quad \text{(III.26)}$$

Let us assume

$$y = m_y + Y,$$

where Y - central random function.

If functions X and Y are not correlated, then the coefficients of statistical linearization $\kappa$ and $\kappa_1$ are as before determined by formulas (III.23) and (III.24), where y it is necessary to replace by $m_y$.

Statistical coefficients of the linearization of the even nonlinearity $F_e(x)$. Let us take the approximating function for $F_e(x)$

in the form

$$z = F_0(x) = z_0 + k_\sigma X, \qquad (III.27)$$

where $z_0$ and $k_\sigma$ are determined with the help of the known formulas (see Chapter II, and also work [34]):

$$z_0 = m_z = \int_{-\infty}^{\infty} F(x)\,p(x)\,dx; \qquad (III.28)$$

$$k_\sigma^{(1)} = \pm\left\{\frac{1}{\sigma_x^2}\int_{-\infty}^{\infty} F^2(x)\,p(x)\,dx - z_0^2\right\}^{\frac{1}{2}}; \qquad (III.29)$$

$$k_\sigma^{(2)} = \frac{1}{\sigma_x^2}\int_{-\infty}^{\infty} F(x)(x - m_x)\,p(x)\,dx. \qquad (III.30)$$

In formulas (III.29)-(III.30) $k_\sigma^{(1)}$ and $k_\sigma^{(2)}$ the statistical coefficients of the linearization of nonlinearity $F_0(x)$ according to random component, calculated respectively using the first and second methods of approximation.

Page 79.

Let us give formulas for values $z_0$, $k_\sigma^{(1)}$ and $k_\sigma^{(2)}$, which correspond to some standard nonlinearity of self-tuning loop, assuming that p(x) - the normal function of density distribution of probability.

For full-wave linear detector $z = k_0|x|$. Therefore under the influence of normal random signal we have

$$z_0 = 2k_\partial\sigma_x\left[\frac{m_x}{\sigma_x}\Phi\left(\frac{m_x}{\sigma_x}\right) + \frac{1}{\sqrt{2\pi}}e^{-\frac{m_x^2}{2\sigma_x^2}}\right];$$
$$k_\sigma^{(1)} = k_\partial\sqrt{1 + \left(\frac{m_x}{\sigma_x}\right)^2 - \frac{z_0^2}{\sigma_x^2 k_\partial^2}};$$
$$k_\sigma^{(2)} = 2k_\partial\Phi\left(\frac{m_x}{\sigma_x}\right).$$

Since function $F_0(x)$ is even, then when $m_x = 0$ $z_0 \neq 0$. For the linear full-wave detector when $m_x = 0$ we have

$$\left.\begin{aligned} z_0 &= \sqrt{\frac{2}{\pi}}k_\partial\sigma_x; \\ k_\sigma^{(1)} &= k_\partial\sqrt{1 - \frac{z_0^2}{\sigma_x^2 k_\partial^2}} = k_\partial\sqrt{\frac{\pi-2}{\pi}}; \\ k_\sigma^{(2)} &= 0. \end{aligned}\right\} \qquad \text{(III.31)}$$

If we as the statistical coefficient of linearization take $k_\sigma^{(3)} = \frac{k_\sigma^{(1)} + k_\sigma^{(2)}}{2}$, then for the nonlinearity in question we will obtain

$$k_\sigma^{(3)} = \frac{1}{2}k_\partial\sqrt{\frac{\pi-2}{\pi}}. \qquad \text{(III.32)}$$

For full-wave square law detector $z = k_\partial x^2$. Then under the influence only of normal random signal we have

$$z_0 = k_\partial\sigma_x^2\left(1 + \frac{m_x^2}{\sigma_x^2}\right);$$
$$k_\sigma^{(1)} = k_\partial\sigma_x\sqrt{2 + 4\frac{m_x^2}{\sigma_x^2}} \cdot \overline{\operatorname{sign}}\, m_x;$$

where

$$k_\sigma^{(2)} = 2k_\partial m_x,$$

$$\overline{\operatorname{sign}} m_x = \begin{cases} +1 \text{ при } m_x \geqslant 0; \\ -1 \text{ при } m_x < 0. \end{cases}$$

Key: (1). with.

Page 80.

When $m_x = 0$ we have

$$m_z = k_\partial \sigma_x^2;$$
$$k_\sigma^{(1)} = \sqrt{2}\, k_\partial \sigma_x;$$
$$k_\sigma^{(2)} = 0.$$

Consequently,

$$k_\sigma^{(3)} = \frac{\sqrt{2}}{2} k_\partial \sigma_x.$$

Let us consider now the case, when the input signal of nonlinearity $F_0(x)$ is the sum of harmonic signal and normal random component:

$$x = A \sin \omega t + x. \qquad (III.33)$$

Since nonlinearity $F_0(x)$ is even, then it is obvious that in the general case the output signal z does not contain the regular component, which has frequency ω. Therefore the approximating function and for the present instance let us take in the form

$$z = \overset{\circ}{z}_0 + \overset{\circ}{k}_\sigma x, \qquad (III.34)$$

where

$$z_0 = m_z = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} F_0(x + A\sin\varphi)\, p(x)\, dx; \qquad \text{(III.35)}$$

$$k_\sigma^\bullet = \frac{1}{2\pi\sigma_x^2}\int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} F_0(x + A\sin\varphi)\, \underset{(\varphi=\omega t)}{xp(x)}\, dx, \qquad \text{(III.36)}$$

Let us note that for the research in general form of the passage of the sum of the harmonic random signal through the nonlinearity of contour of self-adjusting it is necessary to use the generalized method of harmonic linearization[1] together with the method of statistical linearization.

FOOTNOTE [1]. See the book "Methods of harmonic linearization in the design of nonlinear automatic control systems". ("Nonlinear automatic control systems"). M., "Machine building", 1970. ENDFOOTNOTE.

A similar approach makes it possible to take into account the second or higher harmonic at the output of nonlinearity $F_0(x)$. Formula (III.34) corresponds to the simplest and frequently encountered case, when at the output of even nonlinearity it suffices to take into account only constant component (zero harmonic).

Page 81.

Remaining harmonics can be disregarded/neglected, keeping in mind the

filtering properties linear parts of the self-tuning loop. The circumstance indicated usually occurs, if the actuating element of self-tuning loop is inertial component/link. During the research of the self-tuning system it frequently proves to be sufficient to be bounded to the account only of constant component at the output nonlinearity $F_0(x)$; in this case formula (III.34) takes the form

$$z = z_0^{\bullet}, \qquad \text{(III.37)}$$

where $z_0^{\bullet}$ is as before determined by expression (III.35).

Computations according to formulas (III.35) and (III.36) are connected with the fulfillment of sufficiently cumbersome calculations. The information about the methodology of similar computations is contained in works [34, 69].

3. Effect of low-frequency random signal on the nonlinear oscillatory self-tuning systems.

We will be bounded further to the examination of the procedure of calculation of the dynamic properties of nonlinear self-tuning systems, whose fundamental outline works in the mode/conditions of periodic oscillations. Similar systems we will for the brevity call self-oscillating self-tuning systems. Let us assume that the frequencies considered in the spectrum of the random input signal f(t) are arranged/located much lower than frequency of

auto-oscillations or forced oscillations of fundamental outline. The equations of the fundamental outline of the self-tuning system in accordance with the structural scheme, given on Fig. III.3, let us record in the form (with U(t)=0)

$$Q_0(s)\,x + R_0(s)\,F(x,\,y) = S(s)\,f(t), \qquad \text{(III.38)}$$

where $Q_0(s)$, $R_0(s)$ and $S(s)$ - polynomials from the differential operator;

f(t) - random interaction with the known statistical characteristics.

Self-tuning loop is determined by the equations

$$Q_\Phi(s)\,x_\Phi = R_\Phi(s)\,x; \qquad \text{(III.39)}$$

$$z = F_0(x_\Phi); \qquad \text{(III.40)}$$

$$Q_H(s)\,y = R_H(s)\,(z_\vartheta - z). \qquad \text{(III.41)}$$

Equation (III.39) describes the filter of self-tuning loop, and equations (III.40) and (III.41) - respectively nonlinear converter and the actuating element of self-tuning loop. The procedure of calculation of the entire self-tuning system can somewhat be modified depending on the dynamic properties of the single circuit elements of self-adjusting.

Page 82.

Therefore let us describe approach to the research of random

processes in the simplest cases at first, and at the end of the paragraph let us consider more complex problems. Let us assume for the concreteness of the following presentation, that the fundamental outline of the system is found in the mode/conditions of auto-oscillations.

Let the actuating element of self-tuning loop - inertia-free component/link. In the case of the equation of self-tuning system (III.38)in question -(III.41) they will take the form

$$\left.\begin{aligned} Q_0(s)\,x + R_0(s)\,F(x,\ y) &= S(s)\,f(t); \\ Q_\Phi(s)\,x_\Phi &= R_\Phi(s)\,x; \\ z &= F_0(x_\Phi); \\ y &= k_H(z_Э - z). \end{aligned}\right\} \qquad \text{(III.42)}$$

System of equations (III.42) differs from the total system of equations (III.38)-(III.41) in terms of the equation of the actuating elements, where $k_H$ - transmission factor of this device/equipment. Let us assume at first, that the filter in the self-tuning loop is absent, then $Q_\Phi(s) = 1;\ R_\Phi(s) = k_\Phi$. It is not difficult to note that in the case in question the system of equations (III.42) can be reduced to one nonlinear differential equation. Actually/really, assuming/setting $z_Э = 0$ (that it usually occurs for the static actuating elements of self-tuning loop), we will obtain

$$y = -k_H z = -k_H F_0(x_\Phi) = -k_H F_0(k_\Phi x). \qquad \text{(III.43)}$$

As already mentioned above, frequently

$$F(x,\ y)=\lambda(x)\nu(y).$$

Taking into account relationship/ratio (III.43), let us find

$$F(x,\ y)=\lambda(x)\nu[-k_H F_0(k_\Phi x)]=F_1(x).$$

Thus, in the case in question the nonlinear function F(x, y) can be replaced nonlinear of functions $F_1(x)$ from one by the variable x. Usually function $F_1(x)$ is the single-valued odd symmetrical function of the variable x, and therefore research of the nonlinear differential equation

$$Q_0(s)\,x+R_0(s)\,F_1(x)=S(s)\,f(t) \qquad \text{(III.44)}$$

can be carried out with the help of ordinary procedures of the method of statistical linearization, presented in Chapter II of this book. We seek the solution of equation (III.44) in the form

$$x(t)=m_x+A\sin\omega_a t+X(t), \qquad \text{(III.45)}$$

where $m_x$ - constant component;

A - amplitude of auto-oscillations;

$\omega_a$ - frequency of auto-oscillations;

X(t) - central random component.

Page 83.

The statistical linearization of function $F_1(x)$, we further, carrying out divide/mark off equation (III.44) into three single equations for determining constant, sinusoidal and random components.

Let us turn to the analysis of the frequently encountered case, when tuned filter in the self-tuning loop passes only the frequencies, equal or close to the frequency of the auto-oscillations of fundamental outline. In this case self-tuning loop will be opened on random low-frequency component X(t), that contains in signal x(t) and considerably differing in the frequency from the auto-oscillations of fundamental outline. Let us consider the methodology of the solution of this problem in more detail. We will as before seek the solution of system of equations for coordinate x(t) in the form of expression (III.45). In accordance with the second equation of system (III.42) we have

$$x_{\text{ф}} = A_{\text{ф}} \sin(\omega_a t - \varphi), \qquad \text{(III.46)}$$

where

$$A_{\text{ф}} = \left| \frac{R_{\text{ф}}(j\omega_a)}{Q_{\text{ф}}(j\omega_a)} \right| A; \quad \varphi = \arg \left| \frac{R_{\text{ф}}(j\omega_a)}{Q_{\text{ф}}(j\omega_a)} \right|. \qquad \text{(III.47)}$$

Thus, from the latter/last three equations of system (III.42) we will obtain (when $z_э = 0$)

$$y = -k_H z = -k_H F_0(x_\Phi) = -k_H F_0[A_\Phi \sin(\omega_a t - \varphi)|. \quad \text{(III.48)}$$

Consequently, if nonlinearity F(x, y) and $F_0(x_\Phi)$ are determined, for example, by the formulas

$$F(x,\ y) = (h + y)\operatorname{sign} x; \quad F_0(x_\Phi) = |x_\Phi|,$$

then, assuming/setting

$$x_\Phi = kA\sin(\omega_a t - \varphi),$$

when $k_H = -1, k > 0,$ we will obtain

$$F(x,\ y) = (h + kA|\sin(\omega_a t - \varphi)|)\operatorname{sign}(A\sin\omega_a t) =$$
$$= h\operatorname{sign}(A\sin\omega_a t) + kA|\sin(\omega_a t - \varphi)|\operatorname{sign}(A\sin\omega_a t). \quad \text{(III.49)}$$

The coefficient of the harmonic linearization a(A, $\omega_a$) and b(A, $\omega_a$) in this case they depend not only on the amplitude of auto-oscillations A, but also on frequency $\omega_a$. For the determination of the coefficients of harmonic linearization, which correspond to the nonlinearity, determined by formula (III.49), it is possible in certain cases to use the finished expressions, given in work [75].

Let us consider now the cases, when it is necessary to consider the inertness of the actuating element of self-adjusting loop.

Page 84.

Let us assume at first, that the actuating element of the

self-adjusting loop is static inertial component/link, i.e., is described by the equation of the form

$$T\dot{y} = y = k_H (z_э - z). \qquad \text{(III.50)}$$

Let us assume that the filter of self-tuning loop does not pass random component. Since with the inertial actuating element of self-tuning loop two nonlinearity F(x, y) and $F_0(\dot{x}_\Phi)$ are divided by linear part with transfer function $\frac{R_H(s)}{Q_H(s)}$ (see Fig. III.3), the reducing of the nonlinearity indicated to one nonlinearity $F_1(x)$ is here impossible. However, the presence of the inertial component/link between the nonlinearity indicated makes it possible to use in the task in question the method of the separate linearization of the equations of fundamental outline and self-tuning loop[1].

FOOTNOTE [1]. The "Method of harmonic linearization in the design of nonlinear automatic control systems" ("Nonlinear automatic control systems"). M. "Machine building", 1970. ENDFOOTNOTE.

In accordance with the methodology of this method we carry out the harmonic linearization of even nonlinearity $F_0(\dot{x}_\Phi)$ through the constant component, i.e.

$$z = a_{10}(A) A_\Phi, \qquad \text{(III.51)}$$

where $a_{10}(A)$ - the coefficient of the harmonic conversion of the fundamental harmonic into the zero.

In the steady-state mode/conditions from relationships/ratios (III.50) and (III.51) we have

$$y = k_H [z_э - a_{10}(A) A_Ф], \qquad (III.52)$$

where $A_Ф$ is determined through A and $\omega_a$ by formula (III.47). Let us substitute relationship/ratio (III.52) into the formula for F(x, y) and will produce the statistical linearization of this nonlinearity relative to by the variable x. Since usually F(x, y)=λ(x)ν( y), then during the linearization on by the variable x cofactor ν(y) is considered as certain constant value, moreover y is determined by expression (III.52). Further course of solution of the task of the same in question as in the previously described cases. An example of the solution of a problem of the type examined is given in p. 4 of present chapter.

Let us consider the case, when the actuating element of self-tuning loop is the integrating component/link. In this case the system of the differential equations of the self-tuning system is determined by relationships/ratios (III.38)-(III.41), where $Q_H(0) = 0$. We will as before assume that the filter of self-tuning loop does not pass random component, i.e., at the output of filter we have only harmonic component $x_Ф$ in accordance with formulas (III.46) and (III.47).

Page 85.

Solution for coordinate x(t) is determined by formula (III.45), and for coordinate y(t) by the expression

$$y(t) = m_y + Y(t),$$

where Y(t) - central random component. Let us lead the linearization of nonlinearity $F_0(x_\Phi)$ on the constant component in accordance with formula (III.51). The second by the higher harmonics, which are contained in signal z, is disregarded, since they virtually are not passed by the inertial actuating element of self-tuning loop.

It follows from the equation of the actuating element

$$Q_H(s)\,y = sQ_1(s)\,y = k_H(z_э - z); \qquad \text{(III.53)}$$
$$Q_1(0) \neq 0,$$

that in the steady-state mode/conditions with s=0

$$y = \text{const и } z_э = z. \qquad \text{(III.54)}$$

Let us clarify the physical sense of relationships/ratios (III.54). It is known that under the effect of random disturbances on ordinary hunting system the amplitude of auto-oscillations is reduced, and during the considerable disturbances/perturbations stopping the natural oscillations [34, 69] occurs. In the case in

question the actuating element of self-tuning loop in the specific range of a change in the intensity of external disturbances/perturbations provides this value of the varied parameter y, in which $z_э = z$. Since value $z_э$ is proportional to the given value of amplitude $A_э$, the equality $z_э = z$ indicates the maintenance of the given value of the amplitude of auto-oscillations $A = A_э$ (or more accurately - the maintenance of given average/mean value $A_э$). It is obvious that the stabilization of the preset amplitude of auto-oscillations is possible only in the range of the permissible change in the varied parameter y.

From relationship/ratio (III.47), (III.51) and (III.54) it follows that the given value of the amplitude of auto-oscillations $A_э$ is determined by the formulas

$$\left.\begin{aligned} &u_{10}(A_ф)\,A_ф = z_э; \\ &A_э = \frac{A_ф}{|W_ф(i\omega)|}. \end{aligned}\right\} \qquad \text{(III.55)}$$

Let us find the connection/communication between the value of parameter y and the spectral density of random disturbances $S_f(\omega)$.

The passage of the low-frequency random interaction through the self-tuning system can be investigated by different methods. Since periodic component is high-frequency (in comparison with random component), then, applying the principle of separate harmonic

linearization, we will examine coordinate y in the nonlinear function F(x, y) as certain "frozen parameter".

[Page 86.] We will further consider that $m_j = 0$ and, therefore, $m_x = 0$ (since nonlinearity F(x, y) it is assumed to be odd symmetrical relative to x). Let us lead the harmonic linearization of function F(x, y) in accordance with the formulas

$$F(x, y) = F^0 + \left(a + \frac{b}{\omega_a}\right) x, \qquad \text{(III.56)}$$

where

$$\left.\begin{aligned} a &= \frac{1}{\pi A}\int_0^{2\pi} F(A\sin\psi + X,\ A\omega_a\cos\psi,\ y)\sin\psi\, d\psi; \\ b &= \frac{1}{\pi A}\int_0^{2\pi} F(A\sin\psi + X,\ A\omega_a\cos\psi,\ y)\cos\psi\, d\psi \\ & \qquad\qquad (\psi = \omega_a t). \end{aligned}\right\} \qquad \text{(III.57)}$$

It follows from the given formulas that in general/common/total case coefficients F°, a and b are the functions of four values:

$$F^0 = F^0(A,\ X,\ \omega_a,\ y); \quad a = a(A,\ X,\ \omega_a,\ y);$$
$$b = b(A,\ X,\ \omega_a,\ y).$$

Taking into account formula (III.56), let us decompose equation (III.38) of the fundamental outline of the self-tuning system into two equations

$$Q_0(s) X + R_0(s) F^0 = S(s) f(t); \qquad \text{(III.58)}$$
$$Q_0(s) x^* + R_0(s)\left(a + \frac{b}{\omega_a}\right) x^* = 0 \qquad \text{(III.59)}$$

respectively for slowly varying and periodic components. According to

equation (III.58) by the ordinary methods (see Chapter II) it is possible to determine the connection/communication between amplitude A and by slowly varying by the component X, and also the dependence of frequency $\omega_a$ on X. A difference in the relationships/ratios indicated from the analogous relationships/ratios for the ordinary nonlinear systems is dependence A and $\omega_a$ not only on X, but also on parameter y:

$$A = A(X,\ y);\quad \omega_a = \omega_a(X,\ y). \qquad \text{(III.60)}$$

Since in the case $A = A_s = \text{const}$, in question the first of relationships/ratios (III.60) determines the dependence between values X and y. Substituting expressions (III.60) in the relationship/ratio for $F^0(A,\ x,\ \omega_a,\ y)$, we will obtain the dependence

$$F^0 = \Phi^0(X,\ y). \qquad \text{(III.61)}$$

Page 87.

Carrying out the ordinary linearization of characteristic $\Phi^0(X, y)$, we will obtain (for the odd symmetrical relative to x nonlinearity)

$$F^0 = \left(\frac{\partial \Phi^0}{\partial X}\right)_{X=0} X = k_H(y)\,X, \qquad \text{(III.62)}$$

where $k_H(y)$ - coefficient, depending on value y. Consequently, at the low values of the X in comparison with the amplitude auto-oscillations A equation (III.58) can be replacement linear

equation

$$Q_0(s)X + k_H(y)R_0(s)X = S(s)f(t),$$

i.e. with respect to slowly varying interaction f(t) the fundamental outline of the system is characterized by the transfer function

$$\Phi(s, y) = \frac{S(s)}{Q_0(s) + k_H(y)R_0(s)}. \qquad \text{(III.63)}$$

According to the ordinary formula now can be determined the dispersion of slowly varying component $\sigma_x^2$ at the output of the fundamental outline:

$$\sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\Phi(j\omega, y)|^2 S_f(\omega)\, dt, \qquad \text{(III.64)}$$

where $S_f(\omega)$ - the spectral density of random interaction. The value of integral (III.64) can be determined according to tables [69].

The dependence of the average/mean value of the amplitude of auto-oscillations M [A(t)] from dispersion $\sigma_x^2$ under the normal law of distribution X is determined by the formula

$$M[A(t)] = \int_{-\infty}^{\infty} A(x, y)\, p(x)\, dx, \qquad \text{(III.65)}$$

where

$$p(x) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma_x}\right)^2}.$$

Assuming/setting $M[A(t)] = A_a$, we will obtain after integration the relationship/ratio, which relates $A_a, y$ and $\sigma_x$:

$$f(A_a, y, \sigma_x) = 0. \qquad \text{(III.66)}$$

From formulas (III.64) and (III.66) it is possible to determine dependence y and $\sigma_x$ on spectral density $S_f(\omega)$. Thus, the use/application of principle of separate linearization makes it possible to solve the task in question with the help of the same receptions/procedures, as in the case of ordinary nonlinear systems.

Page 88.

Let us assume now, that the level of random disturbances is comparatively great and the direct linearization of the function of bias/displacement $F^\circ = \Phi^\circ(x, y)$ is not admitted. Let us lead the statistical linearization of nonlinearity $F(x, y)$ in accordance with formula (III.21). Substituting relationship/ratio (III.21) in equation (III.38), we will obtain the statistically linearized equation for slowly varying component (when $m_x = 0$)

$$Q_0(s)x + \varkappa_1(A, \sigma_x, y)R_0(s)X = S(s)f(t).$$

The dispersion of random component is determined by the formula

$$\sigma_x^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{S(j\omega)}{Q_0(j\omega) + \varkappa_1(A, \sigma_x, y)R_0(j\omega)}\right| S_f(\omega)\,d\omega. \quad \text{(III.67)}$$

The obtained relationship/ratio can be recorded in the following form:

$$\sigma_x^2 = I_n(A, \sigma_x, y). \quad \text{(III.68)}$$

For harmonic component $x_s$ we have the following equation:

$$[Q_0(s) + \varkappa(A, \sigma_x, y) R_0(s)] x_s = 0, \qquad \text{(III.69)}$$

where $\varkappa(A, \sigma_x, y)$ - statistical coefficient of linearization on the harmonic component.

Amplitude A and frequency $\omega$ with the help of one of the known methods[1] through the coefficient of statistical linearization $\varkappa(A, \sigma_x, y)$ are determined from equation (III.69).

FOOTNOTE [1]. The "Method of harmonic linearization in the design of nonlinear automatic control systems". M., "Machine building", 1970. ENDFOOTNOTE.

Usually for determining the values indicated assume/set $s = j\omega_a$ and equate zero real and imaginary parts of relationship/ratio (III.69):

$$\mathrm{Re}\,[Q_0(j\omega_a) + \varkappa(A, \sigma_x, y) R_0(j\omega_a)] = 0; \qquad \text{(III.70)}$$

$$\mathrm{Im}\,[Q_0(j\omega_a) + \varkappa(A, \sigma_x, y) R_0(j\omega_a)] = 0. \qquad \text{(III.71)}$$

Taking into account that the amplitude of oscillations $A = A_s$ is known and is determined by relationships/ratios (III.55), for determination $\omega_a, \sigma_x$ and y we will have a system of three equations (III.68), (III.70) and (III.71). The equations indicated can be solved

with the help of ordinary graphic receptions/procedures [34, 69]. Frequently $\omega_a$ it is possible to define immediately from equation (III.71) as the function of the parameters of the self-tuning system. Then, solving two equations (III.68) and (III.70) relative to y and $\sigma_x$, we obtain the dependence of the varied parameter y on the rms value of interference at the input of nonlinearity F(x, y). At known value $A_3$ and maximum value $y_{max}$, using the obtained dependence and relationship/ratio (III.67), it is possible to determine the level of external random disturbances, characterized by spectral density $S_f(\omega)$, on which stopping the natural oscillations in the fundamental outline of system occurs, and in the self-tuning loop begins the limitation of value $y = y_{max}$.

Page 89.

4. Examples[1].

FOOTNOTE [1]. This paragraph is written together with V. M. Shpakov. ENDFOOTNOTE.

Example 1. Hunting system with the static self-tuning loop.

Let us consider the effect of stationary random noise on steady state in the self-vibrating self-tuning system with the static

self-tuning loop. The structural scheme of this system is given in Fig. III.3; its behavior is described by system of equations (III.38)-(III.41).

Let us assume that

$$Q_0(s) = (T_1 s + 1)(T_2 s + 1)s;$$
$$R_0(s) = k_1 k_2;$$
$$F(x, y) = (h + y)\operatorname{sign} x;$$
$$F_\Phi(x_\Phi) = x_\Phi \operatorname{sign} x_\Phi;$$
$$S(s) = k_2 (T_1 s + 1).$$

Then system of equations (III.38)-(III.41) takes the form

$$\left.\begin{aligned} &(T_1 s + 1)(T_2 s + 1)sx + k_1 k_2 (h + y)\operatorname{sign} x = k_2 (T_1 s + 1) f(t);\\ &Q_\Phi(s)\, x_\Phi = R_\Phi(s)\, x;\\ &z = x_\Phi \operatorname{sign} x_\Phi;\\ &Q_H(s)\, y = R_H(s)(z_9 - z); \end{aligned}\right\} \quad \text{(III.72)}$$

f(t) - stationary low-frequency random noise.

In the absence of the random disturbance f(t)=0, in the system is established the self-vibrating mode/conditions, whose parameters approximately can be determined with the help of the method of separate harmonic linearization. Assuming/setting

$$x = A_0 \sin \omega_a^* t,$$

where $A_0$ - amplitude;

$\omega_a^*$ - frequency of auto-oscillations with f(t)=0, is realized the harmonic linearization of nonlinearity $F_\Phi(x_\Phi)$ on the zero harmonic in

accordance with formula (III.51) and nonlinearity F(x, y) on the fundamental harmonic, counting y by the parameter. Then, taking into account formulas (III.46), (III.47), instead of system (III.72) we obtain the following system of the linearized equations:

$$\left.\begin{aligned} &(T_1s+1)(T_2s+1)s+k_1k_2a_{11}(A_0, y)=0;\\ &x_\Phi = A_\Phi \sin(\omega_a^* t - \varphi);\\ &z = a_{10}(A_0)A_\Phi;\\ &Q_H(0)y = R_H(0)(z_3 - z), \end{aligned}\right\} \quad \text{(III.73)}$$

where

$$A_\Phi = A_0\left|\frac{R_\Phi(j\omega)}{Q_\Phi(j\omega)}\right|_{\omega=\omega_a} = k_\Phi A_0; \quad \text{(III.74)}$$

$a_{10}(A_0)$ and $a_{11}(A_0, y)$ - the corresponding coefficients of harmonic linearization; $Q_H(s)$ and $R_H(s)$ are replaced on $Q_H(0)$ and $R_H(0)$, since for the steady-state mode/conditions upon consideration only of zero harmonic we have s=0.

Page 90.

Assuming/setting in the first equation of system (III.73) $s=j\omega_a^*$ and equalizing to zero real and imaginary parts of obtained equation, we find the following equations for determining of $A_0$ and $\omega_a^*$:

$$\omega_a^* - T_1T_2\omega_a^{*3} = 0; \quad \text{(III.75)}$$

$$k_1k_2a_{11}(A_0, y) - (T_1+T_2)\omega_a^{*2} = 0. \quad \text{(III.76)}$$

From equation (III.75) we determine

$$\omega_a^* = \frac{1}{\sqrt{T_1T_2}}. \quad \text{(III.77)}$$

Substituting relationship/ratio (III.77) in equation (III.76), we obtain equality for determining of $A_0$:

$$a_{11}(A_0, y) = N, \tag{III.78}$$

where

$$N = \frac{T_1 + T_2}{k_1 k_2 T_1 T_2}. \tag{III.79}$$

For the nonlinearity

$$a_{10} = \frac{2}{\pi}; \tag{III.80}$$

$$a_{11}(A_0, y) = \frac{4(h + y)}{\pi A_0}. \tag{III.81}$$

in question.

Of three latter/last equations (III.73) taking into account relationship/ratio (III.80) we find

$$y = \left(z_3 - \frac{2}{\pi} k_\Phi A_0\right) k_H, \tag{III.82}$$

where

$$k_H = \frac{R_H(0)}{Q_H(0)}.$$

Substituting expressions (III.81) and (III.82) into formula (III.78), we determine

$$A_0 = \frac{4(h + z_3 k_H)}{\pi N + 4\frac{2}{\pi} k_\Phi k_H}. \tag{III.83}$$

In the presence of interference let us represent value x(t) in the form

$$x(t) = m_X + A \sin \omega_a t + X^0(t), \qquad \text{(III.84)}$$

where $m_X$ - constant component;

A - amplitude;

$\omega_a$ - frequency;

$X^0$ - central random component.

Applying the principle of statistical linearization, we will obtain the following expression for the odd symmetrical nonlinear function F(x, y):

$$F(x, y) = (h + y) \operatorname{sign} x = \varkappa_0 m_X + \varkappa A \sin \omega_a t + \varkappa_1 X^0, \qquad \text{(III.85)}$$

where

$\varkappa_0(m_x, A, \sigma_x, y)$, $\varkappa(m_x, A, \sigma_x, y)$ and $\varkappa_1(m_x, A, \sigma_x, y)$ - corresponding coefficients of statistical linearization.

Page 91.

Substituting relationships (III.84) and (III.85) in the first equation of system (III.72) we divide/mark off the obtained equation

into three new ones for determining of constant component, frequency of auto-oscillations and random component:

$$k_1 k_2 \varkappa_0 m_x = k_2 m_f; \tag{III.86}$$

$$(T_1 j\omega + 1)(T_2 j\omega + 1) j\omega + k_1 k_2 \varkappa = 0; \tag{III.87}$$

$$(T_1 s + 1)(T_2 s + 1) s X^0 + k_1 k_2 \varkappa_1 X^0 = k_2 (T_1 s + 1) f^0. \tag{III.88}$$

In these equations $m_f$ - mathematical expectation, and $f^\circ$ - central random component of interference.

Let $m_f = 0$, then, as can be seen from be expression (II.86), $m_x = 0$. In this case

$$\varkappa(0, A, \sigma_x, y) = \frac{h + y}{A} B_0(\alpha); \tag{III.89}$$

$$\varkappa_1(0, A, \sigma_x, y) = \frac{h + y}{\sigma_x} C_0(\alpha), \tag{III.90}$$

where

$$\alpha = \frac{A}{\sigma_x \sqrt{2}}; \; B_0(\alpha) \text{ and } C_0(\alpha)$$

- functions, determined by formulas (III.25) or (III.26), whose graphs are given in works [34, 69].

Let us consider the case encountered in practice, when the spectral density of interference $S_f(\omega)$ is substantially different from zero only in the frequency region, considerably smaller than the frequency of auto-oscillations. In the presence of the filter, tuned to a frequency of auto-oscillations, it is possible to consider that

the interference into the self-tuning loop does not pass, and y is as before determined by expression (III.82), in which $A_0$ it is necessary to replace by A. Substituting expression for y into formulas (III.89) and (III.90), we exclude the dependence $\varkappa$ and $\varkappa_1$ on y:

$$\varkappa(A, \sigma_x) = \frac{h + \left(z_9 - \frac{2}{\pi} k_\Phi A\right) k_H}{a_0} B_0(\alpha); \qquad \text{(III.91)}$$

$$\varkappa_1(A, \sigma_x) = \frac{h + \left(z_9 - \frac{2}{\pi} k_\Phi A\right) k_H}{\sigma_x} C_0(\alpha). \qquad \text{(III.92)}$$

Selecting in equation (III.87) real and imaginary parts, we obtain two equations, from which let us determine the frequency of the auto-oscillations

$$\omega_a = \frac{1}{\sqrt{T_1 T_2}}$$

and the expression, analogous to expression (III.78),

$$\varkappa(A, \sigma_x, y) = N. \qquad \text{(III.93)}$$

As we see, the frequency of auto-oscillations $\omega_a$ coincides with frequency $\omega_a^*$ in the absence of interference. Formula (III.93) establishes the connection/communication between $\sigma_x$ and A. It makes it possible to determine the critical value of value $\sigma_x$, at which the auto-oscillations cease (A=0).

Page 92.

With A=0 for $\varkappa(0, \sigma_x)$ instead of dependence (III.91) we have [34]

$$\varkappa(0, \sigma_x) = \frac{h + z_3 k_H}{\sigma_{x\ cpuea}} \cdot \sqrt{\frac{2}{\pi}}, \tag{III.94}$$

where $\sigma_{x\ cpuea}$ - critical value $\sigma_x$.

Substituting relationship/ratio (III.94) into expression (III.93), we find

$$\sigma_{x\ cpuea} = \frac{h + z_3 k_H}{N} \sqrt{\frac{2}{\pi}}. \tag{III.95}$$

Let the spectral density of interference $S_f(\omega)$ be determined by the expression

$$S_f(\omega) = \frac{2\sigma_f^2 k_\Pi^2 T_\Pi}{\pi\left(1 + T_\Pi^2 \omega^2\right)},$$

where $\sigma_f$ - rms value of interference.

Then from equation (III.88) for the steady-state mode/conditions we obtain

$$\sigma_x^2 = \int_{-\infty}^{\infty} \frac{|k_2(T_1 j\omega + 1)|^2 2\sigma_f^2 k_\Pi^2 T_\Pi}{|[(T_1 j\omega + 1)(T_2 j\omega + 1) j\omega + k_1 k_2 \varkappa_1]|^2 \left[1 + T_\Pi^2 \omega^2\right] \pi} d\omega,$$

or

$$\sigma_x^2 = 4\sigma_f^2 k_\Pi^2 k_2^2 T_\Pi I_4, \tag{III.96}$$

where

$$I_4 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|T_1 j\omega + 1|^2}{|[(T_1 j\omega + 1)(T_2 j\omega + 1) j\omega + k_1 k_2 \varkappa_1]|^2 \left|1 + T_\Pi^2 \omega^2\right|} d\omega;$$

$I_4$ we compute with the help of the tables of integrals [34]:

$$I_4=\frac{c_1b_2+\frac{b_3}{c_4}\left(c_0c_3-c_1c_2\right)}{2\left(c_0c_3^2+c_1^2c_4-c_1c_2c_3\right)}, \qquad \text{(III.97)}$$

where

$$c_0=T_1T_2T_{\Pi}; \qquad c_3=T_{\Pi}k_1k_2\varkappa_1+1;$$
$$c_1=T_1T_2+T_1T_{\Pi}+T_2T_{\Pi}; \qquad c_4=k_1k_2\varkappa_1;$$
$$c_2=T_1+T_2+T_{\Pi}; \qquad b_2=-T_1^2;$$
$$b_3=1.$$

On the strength of the fact that $\varkappa_1$, that depends on A and $\sigma_x$, enters into expression (III.96), this expression is the second equation, which relates unknown A and $\sigma_x$. Consequently, for determining of A and $\sigma_x$ it is necessary to together solve equations (III.93) and (III.96).

Let us assign the following numerical values of the parameters:

$$k_1=k_2=1; \quad T_1=0,75; \quad T_2=1,5; \quad h=3; \quad k_{\Phi}=1;$$
$$k_{H}=10; \quad k_{\Pi}=1; \quad T_{\Pi}=3; \quad z_{Э}=1,25\pi=3,93.$$

Substituting numerical values into expressions (III.79), (III.83) and (III.95), we find N=2, $A_0$=5.33, $\sigma_{x\,\text{сред}}$ =16.7.

Page 93.

We will use the following graphic method of solution. Let us assign several values of A, which lie between $A_0$=5.33 and 0. From equation (III.93) taking into account relationship/ratio (III.91) we obtain the expression

$$B_0(\alpha)=\frac{2A}{42,3-6,37A},$$

with the help of which for each A we compute $B_0(\alpha)$, and then according to graph $B_0(\alpha)$ we find $\alpha$. So as $\alpha=\frac{A}{\sigma_x\sqrt{2}}$, then from those found $\alpha$ we determine $\sigma_x$, to corresponding preset A. As a result on the plane of parameters A and $\sigma_x$ we plot a curve $\sigma_x(A)$, which corresponds to equation (III.93). This curve, depicted in Fig. III.5a, it is noted by digit 1. Then for the obtained values of A and $\sigma_x$ we compute the right side of equation (III.96), which after the substitution of numerical values into the coefficients of expression (III.97) for $I_4$ takes the form

$$\xi^2=12\sigma_l^2\frac{-4,4+\frac{1}{\varkappa_1}[3,375(3\varkappa_1+1)-41,3]}{6,75(3\varkappa_1+1)+124\varkappa_1-82,6(3\varkappa_1+1)}. \qquad \text{(III.98)}$$

For each pair of values A and $\sigma_x$, which correspond to curve 1, through formula (III.92) we find values $\kappa_1$, having preliminarily determined $C_0$ according to formula III.26. Substituting the obtained values $\kappa_1$, having preliminarily determined $C_0$ according to formula III.26. Substituting the obtained values $\kappa_1$ into formula (III.98), we determine the dependence $\xi(A)$, which corresponds to that determined $\sigma_l$. We construct curve $\xi(A)$ on the same plane of parameters A, $\sigma_x$. Point of intersection of its with curve 1 gives the unknown values of

A and $\sigma_x$. Fig. III.5a gives the set of the curves $\xi(A)$, calculated for different values $\sigma_f$. The coordinates of the points of their suppression with curve 1 are given in Table III.1. The values of values y, calculated according to formula (III.82) after determination of A, are there given. Fig. III.5b gives the graphs of thus found dependences $A=A(\sigma_f)$, $\sigma_x=\sigma_x(\sigma_f)$ and $y=y(\sigma_f)$.

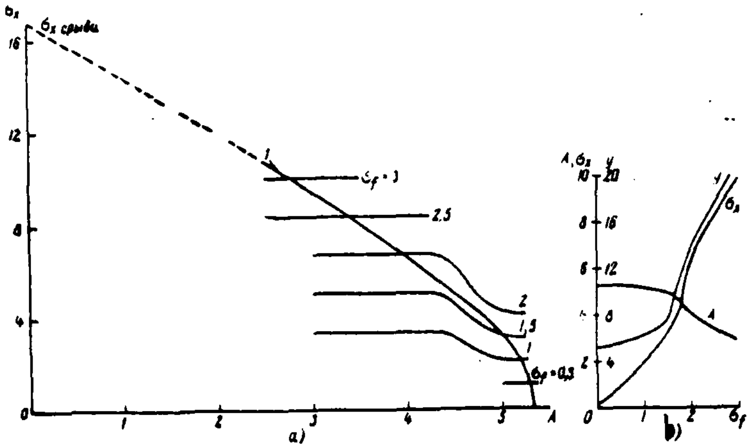


Fig. III.5. Dependence A, y and $\sigma_x$ on $\sigma_f$ for the self-tuning system with the static self-tuning loop.

Page 94.

Example 2. Hunting system with the astatic self-tuning loop.

As the second example let us determine the effect of low-frequency interference on the self-vibrating mode/conditions in the astatic self-tuning system. We will just as in the previous example, assume that the filter of self-tuning loop does not pass random component. The behavior of system is as before described by equations III.72). The presence in the actuating element of integrator makes it possible to record its equation [the fourth in the system (III.72)] in the following form:

$$pQ_{H_1}(s)\,y = R(s)\,(z_э - z).$$

where $Q_{H_1}(s)$ - polynomial, which does not have zero roots.

In this case, as has already been indicated, the interaction of interference leads only to a change in the varied parameter y, while the given value of the amplitude of the auto-oscillations is retained in the system:

$$A_0 = A_3, \tag{III.99}$$

where $A_3$ - given value of amplitude.

Table III.1.

| $\sigma_f$ | 0 | 0,5 | 1,0 | 1,5 | 2,0 | 2,5 | 3,0 |
|---|---|---|---|---|---|---|---|
| $\sigma_x$ | 0 | 1,00 | 2,01 | 3,10 | 6,60 | 8,30 | 9,95 |
| $A$ | 5,33 | 5,29 | 5,19 | 4,97 | 3,97 | 3,38 | 2,79 |
| $y$ | 5,30 | 5,60 | 6,10 | 7,70 | 14,0 | 17,8 | 21,5 |

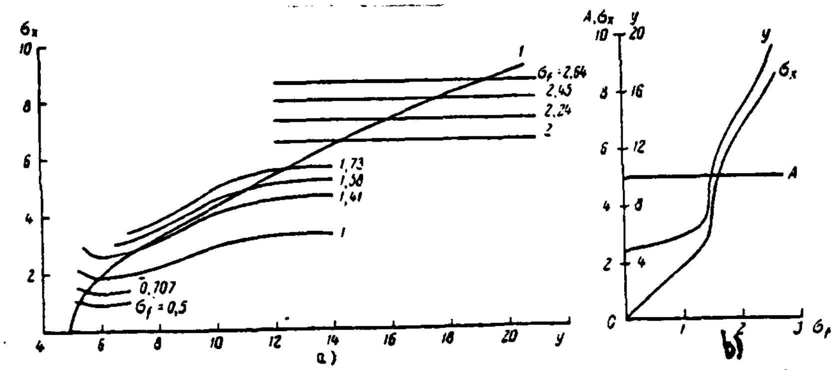



Fig. III.6. Dependence A, y and $\sigma_x$ on $\sigma_f$ for the self-tuning system with the astatic self-tuning loop.

Page 95.

For determining the unknowns y and $\sigma_x$ it is necessary to together solve equations (III.93) and (III.96) taking into account equality (III.99). Let $A_3 = 6$ and the numerical values of the remaining parameters are the same as in the previous example. Substituting dependence (III.81) and (III.99) into equality (III.78), we find value of y in the absence of the interference:

$$y^* = \frac{\pi A_3 N}{4} - h = 4,86.$$

Graphical solution of equations (III.93) and (III.96) is convenient to conduct by the following path. Let us assign the series/row of values y>y*. Substituting in equation (III.93) of equality (III.89) and (III.99), we obtain

$$B_0(\alpha)=\frac{A_sN}{y+h}=\frac{10}{y+3}.$$

From the latter/last relationship/ratio for each value of y we find $B_0$, and then according to graph $B_0(\alpha)$ we determine α. On $\alpha=\frac{A_s}{\sigma_x\sqrt{2}}$ we determine $\sigma_x$. As a result on the plane of parameters y and $\sigma_x$ we plot a curve $\sigma_x=\sigma_x(y)$ (curve 1 in Fig. III.6a). Then according to expression (III.98) we plot a curve ξ=ξ(y) for the given root-mean-square value of interference $\sigma_f$. We compute entering expressions (III.98) coefficient $\varkappa_1$ according to formula (III.90) for the given values of y and in terms of corresponding to them values $\sigma_x$, found from equation (III.93). In Fig. III.6a is given series of curves ξ=ξ(y), constructed for different values $\sigma_f$. The coordinates of the points of intersection of these curves with curve 1 determine the unknown values of y and $\sigma_x$ (Table III.2).

Fig. III.6b depicts curves $y=y(\sigma_f)$ and $\sigma_x=\sigma_x(\sigma_f)$ plotted according

to these values. In the same figure by straight line, parallel to the axis of abscissas, is represented value $A = A_s$.

Fig. III.7 for the comparison gives dependences $\sigma_x = \sigma_x(\sigma_f)$ and $A = A(\sigma_f)$. which occur in ordinary relay system with the same values of the parameters and with the fixed by level relay L=h+y*=7.85, which ensure in the absence of interference the amplitude of auto-oscillations A=5.

The comparison of Fig. III.5b, III.6b and III.7 shows that the self-vibrating self-tuning system with the astatic self-tuning loop, where A=const at all values of the dispersion of interference possesses the greatest freedom from interference $\sigma_f$.

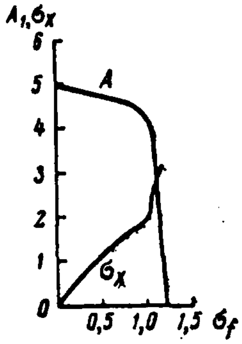



Fig. III.7 Dependence A and $\sigma_x$ on $\sigma_f$ for ordinary hunting system.

Table III.2.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma_f$ | 0,5 | 0,707 | 1,00 | 1,41 | 1,58 | 1,73 | 2,0 | 2,24 | 2,45 | 2,64 |
| $\sigma_x$ | 1,1 | 1,4 | 1,86 | 2,9 | 4,86 | 5,6 | 6,50 | 7,27 | 7,96 | 8,6 |
| $y$ | 5,2 | 5,45 | 5,83 | 7,35 | 11,0 | 12,55 | 14,4 | 15,88 | 17,7 | 19,5 |

Page 96.

It is natural that the stabilization of amplitude indicated, which takes place (in the smaller measure) and for the systems with the static self-tuning loop (see Fig. III.5b), it is possible only in the range of the permissible change in the varied parameter y. It should be noted that an increase in the freedom from interference of self-tuning hunting systems in comparison with ordinary hunting system in the ratio of the amplitude of auto-oscillations is achieved due to the increase of the dispersion of interference $\sigma_x$ at the

output of the self-tuning systems in comparison with the ordinary system.

The latter fact is connected with an increase in the varied parameter y (i.e. limitation level by relay) at increase $\sigma_f$.

Page 97.

## Chapter IV.

## Method of equivalent perturbations.

### 1. Bases of method of equivalent disturbances/perturbations.

Simple representation about the method of equivalent disturbances/perturbations was given in Chapter I. Let us consider the now general theory of mèthod.

The method of equivalent disturbances/perturbations is intended for the approximate determination of the probabilistic characteristics of the output coordinates of nonlinear systems from the preset moments/torques of connection/communication for the input random parameters.

We will as before assume, although this is not compulsory, that the dynamics of the automatic control system being investigated is described by equations (I.1), in which in the case in question the

role of random disturbances $X_r (r = 1, 2, ..., m)$ perform random parameters $V_r$, independent of time t and coordinates $Y_i (i = 1, 2, ...., n)$. Consequently, the equations of the control system can be recorded in the form

$$\frac{dY_1}{dt} = f_1(Y_1, Y_2, \ldots, Y_n, V_1, V_2, \ldots, V_m, t);$$

$$\frac{dY_2}{dt} = f_2(Y_1, Y_2, \ldots, Y_n, V_1, V_2, \ldots, V_m, t);$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{dY_n}{dt} = f_n(Y_1, Y_2, \ldots, Y_n, V_1, V_2, \ldots, V_m, t), \quad \text{(IV.1)}$$

moreover, without breaking generality, it is possible to assume/set

$$\mathrm{M}[V_r] = 0 \qquad (r = 1, 2, \ldots, m).$$

Page 98.

Let us assume also that for parameters $V_r$ there exist and are known the moments/torques of the connection/communication

$$\mu_{r_1 r_2 \ldots r_k} = \mathrm{M}\left[V_{r_1} V_{r_2} \ldots V_{r_k}\right] \qquad \text{(IV.2)}$$
$$(k = 1, 2, \ldots, q; \quad r_1, r_2, \ldots, r_k = 1, 2, \ldots, m).$$

Let further the dynamics of automatic control system be determined by certain set G of the equations, which can be differential, integral, final, etc. Let us as before designate the output coordinates of system through $Y_i (i = 1, 2, ..., n)$. We will examine only one of them, after designating it simply Y.

Task consists of finding of mathematical expectation $u_Y$ of value Y.

The solutions of system of equations G are some functions (generally speaking, nonlinear) of time t and random variables $V_r$:

$$Y = \varphi(t, V_1, V_2, \ldots, V_m).$$

Let us assume that the function $\varphi$ can be expanded in the Maclaurin series in values $V_r$. Being limited by the members of the q degree and by lowering the remainder of resolution, we will obtain

$$Y = \varphi_0 + \sum_{\kappa=1}^{q} \frac{1}{\kappa!} \sum_{r_1=1}^{m} \sum_{r_2=1}^{m} \cdots \sum_{r_\kappa=1}^{m} \left( \frac{\partial^\kappa \varphi}{\partial V_{r_1} \partial V_{r_2} \ldots \partial V_{r_\kappa}} \right) V_{r_1} V_{r_2} \ldots V_{r_\kappa}, \quad \text{(IV.3)}$$

where

$$\varphi_0 = \varphi(t, 0, 0, \ldots, 0). \quad \text{(IV.4)}$$

Index zero in partial derivatives means that they are calculated at point (t, 0, 0, ..., 0).

Page 99.

Passing in equality (IV.3) from the random variables to their mathematical expectations, we will obtain

$$\nu_1 = M[Y] = \varphi_0 + \sum_{\kappa=1}^{q} \frac{1}{\kappa!} \sum_{r_1=1}^{m} \sum_{r_2=1}^{m} \cdots \sum_{r_\kappa=1}^{m} \left( \frac{\partial^\kappa \varphi}{\partial V_{r_1} \partial V_{r_2} \ldots \partial V_{r_\kappa}} \right) \mu_{r_1 r_2 \ldots r_\kappa}. \quad \text{(IV.5)}$$

If we carry out further computations through this formula, then the great difficulties, connected with the need of determining the partial derivatives $\left(\frac{\partial^{\kappa}\varphi}{\partial V_{r_1}\partial V_{r_2}\ldots\partial V_{r_\kappa}}\right)_0$ will arise. The method of equivalent disturbances/perturbations makes it possible to avoid these difficulties.

The essence of method consists of the following.

Let us substitute into the expression for Y particular values $\xi_{rs}$ of parameters $V_r$ and let us lead the expansion of function Y=φ in terms of these parameters. Then analogous with equality (IV.3) we obtain

$$y_s = \varphi_0 + \sum_{\kappa=1}^{q}\frac{1}{\kappa!}\sum_{r_1=1}^{m}\sum_{r_2=1}^{m}\cdots\sum_{r_\kappa=1}^{m}\left(\frac{\partial^{\kappa}\varphi}{\partial V_{r_1}\partial V_{r_2}\ldots\partial V_{r_\kappa}}\right)_0 \xi_{r_1s}\xi_{r_2s}\ldots\xi_{r_\kappa s}. \quad \text{(IV.6)}$$

Values $\xi_{r_\kappa s}$ are called equivalent disturbances/perturbations.

Let us select N different combinations of equivalent disturbances/perturbations $\xi_{r_\kappa s}$ (s=1, 2, ..., N):

$$\begin{array}{l} \xi_{11}, \xi_{21}, \ldots, \xi_{m1}; \\ \xi_{12}, \xi_{22}, \ldots, \xi_{m2}; \\ \cdots\cdots\cdots \\ \xi_{1N}, \xi_{2N}, \ldots, \xi_{mN} \end{array}$$

and let us substitute them into equality (IV.3). We will obtain N equalities of form (IV.6). Multiplying both parts of these equalities by some, as yet not specific, coefficients $\alpha_s (s = 1, 2, \ldots, N)$ and summarizing obtained equations piecemeal, we find

$$S = \sum_{s=1}^{N} \alpha_s y_s = \varphi_0 \sum_{s=1}^{N} \alpha_s + \sum_{\kappa=1}^{q} \frac{1}{\kappa!} \sum_{r_1=1}^{m} \sum_{r_2=1}^{m} \cdots \sum_{r_\kappa=1}^{m} \times$$
$$\times \left( \frac{\partial^\kappa \varphi}{\partial V_{r_1} \partial V_{r_2} \cdots \partial V_{r_\kappa}} \right)_0 \sum_{s=1}^{N} \alpha_s \xi_{r_1 s} \cdots \xi_{r_2 s} \xi_{r_\kappa s}. \qquad \text{(IV.7)}$$

Page 100.

Comparing equalities (IV.5) and (IV.7), we come to the conclusion that sum S will be approximately equal to the mathematical expectation M[Y] of the output coordinate of system, if values $\alpha_s$ and $\xi_{r_\kappa s}$ satisfy the following system of algebraic equations:

$$\sum_{s=1}^{N} \alpha_s = 1; \qquad \text{(IV.8)}$$

$$\sum_{s=1}^{N} \alpha_s \xi_{r_1 s} \xi_{r_2 s} \cdots \xi_{r_\kappa s} = \mu_{r_1 r_2 \ldots r_\kappa} \qquad \text{(IV.9)}$$

$$(\kappa = 1, 2, \ldots, q;\ r_1, r_2, \ldots, r_\kappa = 1, 2, \ldots, m).$$

In fact, choosing as values $\alpha_s$ and $\xi_{r_\kappa s}$ any real solution of system of equations (IV.8) (IV.9), we will obtain

$$S = \sum_{s=1}^{N} \alpha_s y_s = M[Y]. \qquad \text{(IV.10)}$$

Values $y_s$ can be computed by the method of solution of the reference system of equations G with the substitution into this system of corresponding equivalent disturbances/perturbations $\xi_{r_\kappa s}$ instead of random parameters $V_r$. In all, thus, it is necessary to fulfill N solutions of system G, each time substituting new values $\xi_{r_\kappa s}$ of parameters $V_r$. With the help of this reception/procedure it is possible to avoid the computation of derivatives $\left(\frac{\partial^\kappa \varphi}{\partial V_{r_1} \partial V_{r_2} \ldots \partial V_{r_\kappa}}\right)_0$, it is essential to reduce the laboriousness for computations and to raise their accuracy.

In work [34] it is shown that for

$$N = C^q_{m+q} \qquad \text{(IV.11)}$$

the system of equations (IV.8), (IV.9) during the proper selection of values $\xi_{r_\kappa s}$ has real solution. However, in this case number N will be, as a rule, very considerable; therefore for determining the mathematical expectation of coordinate Y from formula (IV.10) a large number of solutions of system G will be required.

During the practical computations it is expedient in each specific case depending on a number of the random variables m considered and the degree q of the approximating polynomial to select this set of equivalent disturbances/perturbations $\xi_{r_\kappa s}$, in order as far

as possible a larger number of coefficients $a_s$ in formula (IV.10) to become zero. In this case will be substantially reduced a number of necessary solutions of system G.

Some examples of the selection of values $\xi_{r_k s}$ will be given below. Now let us examine how for value Y it is possible to compute the moments/torques of external orders.

Page 101.

For the definition of the central multipole moments, as is known, sufficient to find the appropriate initial moments/torques, since between them and central moments there are single bonds. Therefore we will be bounded only to indication of that, in what way it is possible to find initial moments. Let us designate the initial moment of order p of value Y through $\nu_p$:

$$\nu_p = M[Y^p]. \qquad \text{(IV.12)}$$

Let us designate further

$$Y^p = \varphi^p(t, V_1, V_2, \ldots, V_m) = \psi(t, V_1, V_2, \ldots, V_m). \quad \text{(IV.13)}$$

Relative to function $\psi$ completely remain valid all reasonings, which were carried out for the function $\varphi$. Therefore, expanding function $\psi$ in Maclaurin series on $V_r$ and passing to the mathematical

expectations, we will obtain the expression, analogous to expression (IV.5), with the only difference that the in it instead of the function $\varphi$ will figure $\psi$:

$$\nu_p = M[Y^p] = \psi_0 + \sum_{\kappa=1}^{q} \frac{1}{k!} \sum_{r_1=1}^{m} \sum_{r_2=1}^{m} \sum_{r_\kappa=1}^{m} \times \times \left( \frac{\partial^\kappa \psi}{\partial V_{r_1} \partial V_{r_2} \ldots \partial V_{r_\kappa}} \right)_0 \mu_{r_1 r_2 \ldots r_\kappa}. \quad \text{(IV.14)}$$

Substituting in equality (IV.13) for parameters $V_r$ previously selected values $\xi_{r_\kappa s}$ and expanding function $\psi$ in Maclaurin series in these values, we will obtain

$$y_s^p = \psi_0 + \sum_{\kappa=1}^{q} \frac{1}{\kappa!} \sum_{r_1=1}^{m} \sum_{r_2=1}^{m} \cdots \sum_{r_\kappa=1}^{m} \left( \frac{\partial^\kappa \psi}{\partial V_{r_1} \partial V_{r_2} \ldots \partial V_{r_\kappa}} \right)_0 \times \times \xi_{r_1 s} \xi_{r_2 s} \ldots \xi_{r_\kappa s}. \quad \text{(IV.15)}$$

After multiplying the further obtained equations on summarizing by s and taking into account of equation (IV.8) and (IV.9), let us record

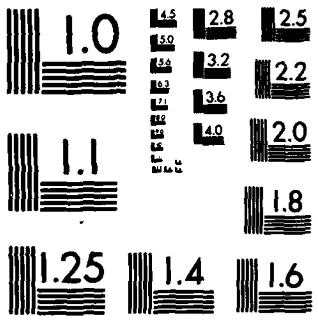
$$\nu_p = M[Y^p] = \sum_{s=1}^{N} \alpha_s y_s^p. \quad \text{(IV.16)}$$

Consequently, for determining the moment/torque $\nu_p$ it suffices to compute the p degrees previously obtained solutions $y_s$ and result to substitute into formula (IV.16). Analogously can be found any moments of connection/communication for functions $Y_1$, $Y_2$, ..., [34].

With comparatively small Mach numbers and q, as this follows from formula (IV.11), the method of equivalent

AD-A143 297 STATIC METHODS IN THE DESIGN OF NONLINEAR AUTOMATIC CONTROL SYSTEMS(U) FOREIGN TECHNOLOGY DIV WRIGHT-PATTERSON AFB OH N I ANDREYEV ET AL. 27 JUN 84 3/9

UNCLASSIFIED FTD-ID(RS)T-1734-83 F/G 12/1 NL




MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

disturbances/perturbations proves to be sufficiently simple and economical from the point of view of the volume of computational work.

Page 102.

Unfortunately, a straight/direct attempt at the use/application of a method with a large (order of tens and hundreds) number of random parameters, which affect the dynamics of system, leads to the such complicated computations, that the use of a method of statistical simulation proves to be more appropriate. The space of computations in the implementation of method sharply grows also with the refinement of hypothesis about degree of q of the polynomial, which approximates the dependence of the coordinate Y of system being investigated on random parameters $V_r$. The complexity of the application of the method of equivalent disturbances/perturbations grows also upon consideration of the connections/communications, which exist between the random parameters, and also due to the heterogeneity of the laws of their distribution. Therefore during the derivation of calculation formulas for determining the probabilistic characteristics of coordinate Y is expediently to preliminarily convert the system of random variables $V_1, V_2, \ldots, V_m$ to the system of independent variables $W_1, W_2, \ldots, W_m$, subordinated to the preset law of distribution q(w), identical for all $W_j (j = 1, 2, \ldots, m)$.

2. Algorithm of the reduction of the system of the dependent random variables to the system of independent variables.

1. Formulation of problem. Let the system of dependent random variables $V_1, V_2, \ldots, V_m$ be preset by the differential law of distribution

$$p = p(v_1, v_2, \ldots, v_m). \qquad \text{(IV.17)}$$

It is necessary to find this conversion

$$W_i = F_i(V_1, V_2, \ldots, V_m) \qquad (i = 1, 2, \ldots, m), \qquad \text{(IV.18)}$$

so that values $W_1, W_2, \ldots, W_m$ obtained in this case would be independent and allocations

$$q_i = q_i(w_i) \qquad (i = 1, 2, \ldots, m), \qquad \text{(IV.19)}$$

were subordinated to the given laws, i.e. the differential law q of distribution of the system of values $W_1, W_2, \ldots, W_m$ must take the form

$$q(w_1, w_2, \ldots, w_m) = \prod_{i=1}^{m} q_i(w_i). \qquad \text{(IV.20)}$$

2. Solution of problem with m=2. Let us consider the preliminarily simplest case of the transformation of system of two dependent variables $V_1$, $V_2$, subordinated to the law of distribution $p(v_1, v_2)$, to independent variables $W_1$ and $W_2$, subordinated to the law of distribution

$$q(w_1, w_2) = q_1(w_1)\, q_2(w_2). \qquad \text{(IV.21)}$$

Page 103.

Let us select transformation (IV.18) as follows:

$$\left.\begin{array}{c} W_1 = V_1; \\ W_2 = F_2(V_1, V_2). \end{array}\right\} \qquad \text{(IV.22)}$$

Let us introduce now into the examination two planes: plane $V_1OV_2$ of initial random variables and plane $W_1\Omega W_2$ of the converted values (Fig. IV.1). Let us take in plane $V_1OV_2$ the infinitesimally narrow band $\Sigma$, parallel to axis $OV_2$ and limited by the straight lines

$$V_1 = v_1; \quad V_1 = v_1 + dv_1;$$
$$V_2 = -\infty, \quad V_2 = v_2.$$

In accordance with expressions (IV.22) in plane $W_1\Omega W_2$ the infinitesimally narrow band $\Sigma'$, parallel to axis $\Omega W_2$ and limited by the straight lines

$$W_1 = V_1 = v_1; \quad W_1 = v_1 + dv_1;$$
$$W_2 = -\infty; \quad W_2 = F(v_1, v_2) = w_2$$

will be obtained also.

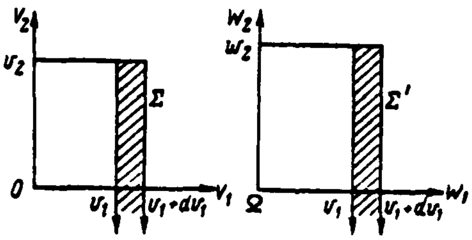
We will assume that together with transformation (IV.22) there exists and inverse transformation:

$$\left.\begin{array}{c} V_1 = W_1; \\ V_2 = G(W_1, W_2), \end{array}\right\} \qquad \text{(IV.23)}$$

i.e. between values ($V_1$, $V_2$) and ($W_1$, $W_2$) is established/installed one-to-one conformity. In this assumption to each point of band Σ will correspond one and only one point of band Σ' and vice versa. Therefore the probabilities of falling of points ($V_1$, $V_2$) and ($W_1$, $W_2$) respectively in bands Σ and Σ' will be equal. On this foundation and taking into account expression (IV.21) it is possible to record

$$\int_{-\infty}^{v_2} [p(v_1, v_2)\,dv_1]\,dv_2 = \int_{-\infty}^{w_2} [q(w_1, w_2)\,dv_1]\,dw_2 =$$

$$= \int_{-\infty}^{w_2} [q(v_1, w_2)\,dv_1]\,dw_2 = \int_{-\infty}^{w_2} [q_1(v_1)\,q_2(v_2)\,dv_1]\,dw_2.$$




Transformation of the system of two values.

Page 104.

Hence

$$\int_{-\infty}^{v_2} p(v_1, v_2)\,dv_2 = q_1(v_1)\int_{-\infty}^{w_2} q_2(w_2)\,dw_2. \qquad \text{(IV.24)}$$

Under the condition

$$q_2(w_2) > 0 \qquad \text{(IV.25)}$$

the function

$$Q_2(w_2) = \int_{-\infty}^{w_2} q_2(w_2)\,dw_2 \qquad \text{(IV.26)}$$

will be monotonically increasing.

It is analogous with the observance of the condition

$$p(v_1, v_2) > 0 \qquad \text{(IV.27)}$$

the function

$$P_2(v_1, v_2) = \int_{-\infty}^{v_2} p(v_1, v_2)\,dv_2 \qquad \text{(IV.28)}$$

will be monotonically increasing relative to argument $v_2$. Therefore there are inverse functions $Q_2^{-1}$ and $P_2^{-1}$. Consequently, equality (IV.24) can be solved both relative to $v_2$ and relative to $w_2$, i.e., satisfaction of conditions (IV.25) and (IV.27) ensures the one-to-one correspondence of transformations (IV.22) and (IV.23). After solving equality (IV.24) relative to $w_2$ and after taking into consideration of expression (IV.26) and (IV.28), let us record

$$w_2 = Q_2^{-1}\left[\frac{P_2(v_1, v_2)}{q_1(v_1)}\right]. \qquad \text{(IV.29)}$$

Since during transformation (IV.22) the law of distribution of value $V_1$ is retained, since $W_1=V_1$,

$$q_1(v_1) = p_1(v_1),$$

where

$$p_1(v_1) = \int_{-\infty}^{+\infty} p(v_1, v_2)\, dv_2 = P_2(v_1, +\infty).$$

Consequently, formula (IV.29) can be represented in this final form:

$$w_2 = Q_2^{-1}\left[\frac{P_2(v_1, v_2)}{P_2(v_1, +\infty)}\right], \qquad \text{(IV.30)}$$

i.e. function $F(V_1, V_2)$ in the transformation (IV. 22) in this case is determined by the expression

$$F_2(V_1, V_2) = Q_2^{-1}\left[\frac{P_2(V_1, V_2)}{P_2(V_1, +\infty)}\right]. \qquad \text{(IV.31)}$$

where $P_2(V_1, V_2)$ is found by formula (IV.28) as a result of substitution into it $v_1=V_1$ and $v_2=V_2$.

Page 105.

Example. The system of the random variables $V_1$, $V_2$ is subordinated to the differential law of distribution

$$p(v_1, v_2) = \frac{1}{2\pi |v_1|} e^{-\frac{1}{2}\left(v_1^2 + \frac{v_2^2}{v_1^2}\right)}.$$

Let us find the transformation, which leads values $V_1$, $V_2$ to the system of normally distributed independent variables $W_1$, $W_2$.

On the condition of the task

$$Q_2(w_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{w_2} e^{-\frac{w_2^2}{2}} dw_2.$$

On the other hand, according to formula (IV.28) we have

$$P_2(v_1, v_2) = \int_{-\infty}^{v_2} \frac{1}{2\pi |v_1|} e^{-\frac{1}{2}\left(v_1^2 + \frac{v_2^2}{v_1^2}\right)} dv_2 = \frac{1}{2\pi} e^{-\frac{v_1^2}{2}} \int_{-\infty}^{\frac{v_2}{|v_1|}} e^{-\frac{z^2}{2}} dz$$

and

$$P_2(v_1, +\infty) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v_1^2}{2}}.$$

On the basis of formula (IV.31) taking into account the fact that $F_2=w_2$, it is possible record

$$Q_2(w_2)=\frac{P_2(v_1, v_2)}{P_2(v_1, +\infty)}.$$

After substituting the here obtained above expressions for $Q_2$ and $P_2$, let us find

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{w_2} e^{-\frac{w_2^2}{2}}dw_2=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{v_2}{|v_1|}} e^{-\frac{z^2}{2}}dz.$$

Hence we come to the conclusion that

$$w_2=\frac{v_2}{|v_1|},$$

i.e.

$$v_1=w_1;$$
$$v_2=w_2|w_1|.$$

Page 106.

3. Solution of problem with arbitrary m. The solution of problem with arbitrary m can be obtained by the simple generalization of solution, found for m=2. In fact, let us select transformations (IV.18) in the form

$$\left.\begin{aligned} W_1&=V_1;\\ W_2&=V_2;\\ &\dots\dots\\ W_{m-1}&=V_{m-1};\\ W_m&=F_m(V_1, V_2, \dots, V_m),\end{aligned}\right\}\qquad(IV.32)$$

assuming that function $F_m$ monotonically increasing on $V_m$. Then there is inverse transformation:

$$\left.\begin{array}{c} V_1 = W_1; \\ V_2 = W_2; \\ \cdot\ \cdot\ \cdot\ \cdot \\ V_{m-1} = W_{m-1}; \\ V_m = G(W_1,\ W_2,\ \ldots,\ W_m). \end{array}\right\} \qquad (\mathrm{IV.33})$$

Let us take in the m-dimensional parameter spaces $V_1, V_2, \ldots, V_m$ infinite narrow rectangular prism Σ, parallel to axis $OV_m$ and limited by the planes

$$\begin{array}{c} V_1 = v_1;\quad V_1 = v_1 + dv_1; \\ V_2 = v_2;\quad V_2 = v_2 + dv_2; \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ V_{m-1} = v_{m-1};\quad V_{m-1} = v_{m-1} + dv_{m-1}; \\ V_m = -\infty;\quad V_m = v_m. \end{array}$$

In accordance with expressions (IV.32) when there is inverse transformation (IV.33), to prism Σ in the parameter spaces $W_1, W_2, \ldots, W_m$ will correspond the prism Σ', limited by the planes

$$\begin{array}{ll} W_1 = V_1 = v_1; & W_1 = v_1 + dv_1; \\ W_2 = V_2 = v_2; & W_2 = v_2 + dv_2; \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ W_{m-1} = V_{m-1} = v_{m-1} & W_{m-1} = v_{m-1} + dv_{m-1}; \\ W_m = -\infty; & W_m = F_m(v_1,\ v_2,\ \ldots,\ v_m) = w_m. \end{array}$$

Page 106.

As a result of one-to-one conformity of points $(V_1, V_2, \ldots, V_m)$ and $(W_1, W_2, \ldots, W_m)$ are equal to probability their entry respectively into the limits of prisms Σ and Σ', i.e., it is possible to record

$$\int_{-\infty}^{v_m} [p(v_1, v_2, \ldots, v_m)\, dv_1\, dv_2 \ldots dv_{m-1}]\, dv_m =$$
$$= \int_{-\infty}^{w_m} [q(w_1, w_2, \ldots, w_m)\, dw_1, dw_2 \ldots dw_{m-1}]\, dw_m.$$

Taking into account the relationships/ratios

$$\left.\begin{array}{ll} v_1 = w_1; & dv_1 = dw_1; \\ v_2 = w_2; & dv_2 = dw_2; \\ \cdots & \cdots \\ v_{m-1} = w_{m-1}; & dv_{m-1} = dw_{m-1}; \end{array}\right| \qquad \text{(IV.34)}$$

let us record

$$\int_{-\infty}^{v_m} p(v_1, v_2, \ldots, v_m)\, dv_m = \int_{-\infty}^{w_m} q(v_1, v_2, \ldots, v_{m-1}, w_m)\, dw_m. \qquad \text{(IV.35)}$$

After substituting here expression for $q(v_1, v_2, \ldots, v_m)$ from equality (IV.20), we will obtain

$$\int_{-\infty}^{v_m} p(v_1, v_2, \ldots, v_m)\, dv_m = q_{m-1}(v_1, v_2, \ldots, v_{m-1}) \times \int_{-\infty}^{w_m} q_m(w_m)\, dw_m, \qquad \text{(IV.36)}$$

where $q_{m-1}(v_1, v_2, \ldots, v_{m-1})$ - differential law of distribution of the system of values $V_1, V_2, \ldots, V_{m-1}$.

As a result of the fulfillment of equalities (IV.34) the law of distribution of values $V_1, V_2, \ldots, V_{m-1}$ coincides with the law of distribution of values $W_1, W_2, \ldots, W_{m-1}$, therefore it is possible to record

$$q_{m-1}(v_1, v_2, \ldots, v_{m-1}) = p_{m-1}(v_1, v_2, \ldots, v_{m-1}) = \int_{-\infty}^{+\infty} p(v_1, v_2, \ldots, v_m)\, dv_m. \qquad \text{(IV.37)}$$

When

$$q_m(w_m) > 0, \qquad \text{(IV.38)}$$

the function

$$Q_m(w_m) \quad \int_{-\infty}^{w_m} q_m(w_m)\, dw_m \qquad \text{(IV.39)}$$

will be monotonically increasing.

Page 108.

Analogously under the condition

$$p(v_1, v_2, \ldots, v_m) > 0 \qquad \text{(IV.40)}$$

the function

$$P_m(v_1, v_2, \ldots, v_m) = \int_{-\infty}^{v_m} p(v_1, v_2, \ldots, v_m)\, dv_m \qquad \text{(IV.41)}$$

will be also monotonically increasing relative to $v_m$. Therefore there are inverse functions $Q_m^{-1}$ and $P_m^{-1}$. Hence it follows that equality

(IV.36) can be solved both relative to $v_m$, and relative to $w_m$, i.e., satisfaction of conditions (IV.38) and (IV.40) ensures the one-to-one correspondence of transformations (IV.32) and (IV.33).

Solving equality (IV.36) relative to $w_m$ and taking into account expression (IV.37), (IV.39) and (IV.41), let us record

$$w_m = Q_m^{-1}\left[\frac{p_m(v_1, v_2, \ldots, v_m)}{p_{m-1}(v_1, v_2, \ldots, v_{m-1})}\right] = Q_m^{-1}\left[\frac{P_m(v_1, v_2, \ldots, v_{m-1}, v_m)}{P_m(v_1, v_2, \ldots, v_{m-1}, \infty)}\right], \tag{IV.42}$$

i.e. function $F_m(V_1, V_2, \ldots, V_m)$ in transformation (IV.32) is determined by the expression

$$W_m = F_m(V_1, V_2, \ldots, V_m) - Q_m^{-1}\left[\frac{P_m(V_1, V_2, \ldots, V_{m-1}V_m)}{P_m(V_1, V_2, \ldots, V_{m-1}, \infty)}\right]. \tag{IV.43}$$

As a result of the single use/application of transformation (IV.32) and (IV.43) will be obtained the new system of random variables $W_1, W_2, \ldots, W_m$, in which value $W_m$ is not dependent on previous m-1 values $W_i (i = 1, 2, \ldots, m-1)$ and by the subordinate to the preset law of distribution $q_m(w_m)$. After m steps/pitches, from which latter/last step/pitch will be the simple transformation of the random variable $V_1$ toward the required distribution law, instead of initial m of values $V_1, V_2, \ldots, V_m$ will be obtained new m of the values, independent in their set and subordinated to the preset laws of distribution $q_i$.

Example. The system of values $V_1$, $V_2$, $V_3$ is subordinated to the

law of distribution of the form

$$p(v_1, v_2, v_3) = \frac{1}{(\sqrt{2\pi})^3 |v_2 - v_1|} e^{-\frac{1}{2}\left(2v_1^2 - 2v_1 v_2 + v_2^2 + \frac{v_3^2}{(v_2 - v_1)^2}\right)}.$$

To find the transformation, which leads values $V_1$, $V_2$, $V_3$ to the system of independent variables, subordinated to the normal laws of distribution of the form

$$q_i(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} \quad (i = 1, 2, 3). \qquad \text{(IV.44)}$$

Page 109.

I step. From formula (IV.41) let us find

$$P_3(v_1, v_2, v_3) = \int_{-\infty}^{v_3} p(v_1, v_2, v_3)\, dv_3 = \frac{1}{(\sqrt{2\pi})^3 |v_2 - v_1|} e^{-\frac{1}{2}\left(2v_1^2 - 2v_1 v_2 + v_2^2\right)} \times$$
$$\times \int_{-\infty}^{v_3} e^{-\frac{v_3^2}{(v_2 - v_1)^2}} dv_3.$$

After fulfilling in the integral the replacement of variable according to the formula

$$\xi = \frac{v_3}{|v_2 - v_1|},$$

we will obtain

$$P_3(v_1, v_2, v_3) = \frac{1}{(\sqrt{2\pi})^3} e^{-\frac{1}{2}\left(2v_1^2 - 2v_1 v_2 + v_2^2\right)} \int_{-\infty}^{\frac{v_3}{|v_2 - v_1|}} e^{-\frac{\xi^2}{2}} d\xi.$$

Hence

$$P_3(v_1, v_2, \infty) = \frac{1}{2\pi} e^{-\frac{1}{2}\left(2v_1^2 - 2v_1 v_2 + v_2^2\right)}.$$

Let us further record expression for function (IV.39), taking into account that on the condition of an example the converted values must be subordinated to the normal law of distribution of form (IV.44):

$$Q_3(w_3) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{w_3} e^{-\frac{w^2}{2}} dw.$$

Then on the basis of equality (IV.42) it is possible to record

$$Q_3(w_3) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{w_3} e^{-\frac{w^2}{2}} dw = \frac{P_3(v_1, v_2, v_3)}{P_3(v_1, v_2, \infty)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{v_3}{|v_3 - v_1|}} e^{-\frac{\xi^2}{2}} d\xi.$$

Consequently,

$$w_3 = \frac{v_3}{|v_3 - v_1|}.$$

The differential law $p_2(v_1, v_2)$ allocation of two remaining untransformed/unconverted values $V_1$, $V_2$ will, obviously, coincide with function $P_3(v_1, v_2, \infty)$, i.e.

$$p_2(v_1, v_2) = P_3(v_1, v_2, \infty) = \frac{1}{2\pi} e^{-\frac{1}{2}\left(2v_1^2 - 2v_1 v_2 + v_2^2\right)}.$$

II step. We convert the system of values $V_1$, $V_2$ to the new values $W_1$, $W_2$, applying the algorithm, analogous to algorithm of the

I step. The solution of problem in the principle does not differ from that presented in an example. After fulfilling the necessary computations, let us find

$$w_2 = v_2 - v_1;$$
$$w_1 = v_1.$$

Page 110.

Thus, the system of the equalities, which connect values $V_1$, $V_2$, $V_3$ with the converted values, in this case takes the form

$$V_1 = W_1;$$
$$V_2 = W_2 + W_1;$$
$$V_3 = |W_2| \cdot W_3.$$

3. Determination of the probabilistic characteristics of the output coordinate of nonlinear system for case of q=2.

Let us consider the use/application of a method of equivalent disturbances/perturbations for the case, when in expansion (IV.3) it suffices to take into account only second-order quantities relative to random parameters $V_r$. In contrast to the simplest linearization of the output coordinates of system from parameters $V_r$ this quadratic approximation makes it possible to frequently obtain the essential refinement of results.

The especially simple solution of problem can be obtained, if to preliminarily lead parameters $V_r$ to canonical system [77], i.e., to attain their noncorrelation. We will assume that this stage is carried out. Then

$$M[V_{r_1}V_{r_2}] = 0$$

with $r_1 \neq r_2$ and

$$M[V_rV_r] = M[V_r^2] = \sigma_r^2.$$

Equations (IV.8) and (IV.9) for q=2 take the form

$$\sum_{s=1}^{N} \alpha_s = 1; \qquad \text{(IV.45)}$$

$$\sum_{s=1}^{N} \alpha_s \xi_{rs} = 0 \qquad (r = 1, 2, \ldots, m); \qquad \text{(IV.46)}$$

$$\sum_{s=1}^{N} \alpha_s \xi_{r_1 s} \xi_{r_2 s} = 0 \qquad (r_1 \neq r_2;\ r_1, r_2 = 1, 2, \ldots, m); \qquad \text{(IV.47)}$$

$$\sum_{s=1}^{N} \alpha_s \xi_{rs}^2 = \sigma_r^2 \qquad (r = 1, 2, \ldots, m). \qquad \text{(IV.48)}$$

A number of different equations is determined in the general case according to formula (IV.11):

$$N = C_{m+2}^{q} = \frac{(m+2)(m+1)}{2}. \qquad \text{(IV.49)}$$

Page 111.

This number rapidly grows with increase of m. However, by the special selection of equivalent disturbances/perturbations $\xi_{rs}$ and

$\xi_{r}$, it is possible to attain rotation/access into many zero coefficients $\alpha_s$ and it is essential to reduce the number of necessary solutions of equations G of the investigated nonlinear automatic control system thereby. Let us select, for example, value $\xi_r$ in accordance with Tables IV.1.

In this case of N=m+2, which (especially with the larger values m) is substantially less than the number of solutions of system G, determined from formula (IV.49). Equations (IV.45) and (IV.46) in this case are reduced to the form

$$\sum_{s=1}^{m+2} \alpha_s = 1; \qquad \text{(IV.50)}$$

$$\alpha_r + \alpha_{m+1} - \alpha_{m+2} - 0 \qquad \text{(IV.51)}$$
$$(r = 1, 2, \ldots, m)$$

Summarizing all m of equations (IV.51) and taking into account expression (IV.50), let us find

$$\sum_{s=1}^{m+2} \alpha_s + (m-1)\alpha_{m+1} - (m+1)\alpha_{m+2} = 0;$$

hence

$$(m+1)\alpha_{m+2} = (m-1)\alpha_{m+1} + 1. \qquad \text{(IV.52)}$$

On the basis of data of Table IV.1 equation (IV.47) and (IV.48) they are reduced to this form:

$$\alpha_{m+1} + \alpha_{m+2} = 0, \qquad \text{(IV.53)}$$

$$(\alpha_r + \alpha_{m+1} + \alpha_{m+2})\xi_r^2 = \sigma_r^2. \qquad \text{(IV.54)}$$

From the system of equations (IV.52) and (IV.53) let us find

$$\left.\begin{aligned} \alpha_{m+1} &= -\frac{1}{2m}; \\ \alpha_{m+2} &= \frac{1}{2m}. \end{aligned}\right\} \qquad \text{(IV.55)}$$

Table IV.1.

| $s$ | $r$: 1 | 2 | 3 | ....... | $m-2$ | $m-1$ | $m$ |
|---|---|---|---|---|---|---|---|
| 1 | $\xi_1$ | 0 | 0 | ...... | 0 | 0 | 0 |
| 2 | 0 | $\xi_2$ | 0 | ...... | 0 | 0 | 0 |
| ... | ... | ... | ... | ... | ... | ... | ... |
| $m-1$ | 0 | 0 | 0 | ...... | 0 | $\xi_{m-1}$ | 0 |
| $m$ | 0 | 0 | 0 | ...... | 0 | 0 | $\xi_m$ |
| $m+1$ | $\xi_1$ | $\xi_2$ | $\xi_3$ | ...... | $\xi_{m-2}$ | $\xi_{m-1}$ | $\xi_m$ |
| $m+2$ | $-\xi_1$ | $-\xi_2$ | $-\xi_3$ | ...... | $-\xi_{m-2}$ | $-\xi_{m-1}$ | $-\xi_m$ |

Page 112.

Then on the basis of expression (IV.51) we obtain

$$\alpha_r = \frac{1}{m} \quad (r = 1, 2, \ldots, m). \qquad \text{(IV.56)}$$

Finally, from equalities (IV.53), (IV.54) and (IV.56) we find the unknown equivalent disturbances/perturbations

$$\xi_r = \sigma_r \sqrt{m}.$$

Let us designate through $y_s$ the solutions of system G, which are obtained with the substitution instead of random parameters $V_r$ of nonrandom values $\xi_r$ according to Tables IV.1.

Then on the basis of formulas (IV.10), (IV.16), (IV.55) and (IV.56) it is possible to record

$$M[Y] = \sum_{s=1}^{m+2} \alpha_s y_s = \frac{1}{m}\left(\sum_{s=1}^{m} y_s + \frac{y_{m+2} - y_{m+1}}{2}\right); \qquad (IV.57)$$

$$D[Y] = M[Y^2] - (M[Y])^2 = \frac{1}{m}\left(\sum_{s=1}^{m} y_s^2 + \frac{y_{m+2}^2 - y_{m+1}^2}{2}\right) - (M[Y])^2. \qquad (IV.58)$$

4. Determination of the probabilistic characteristics of the output coordinate of nonlinear system for case of q=3.

Let us assume that during the determination of mathematical expectation and dispersion of the output coordinate Y of nonlinear automatic control system it is necessary to take into account the effect of the small third-order quantities relative to parameters Let us assume that values $V_r$ are not mutually connected and have the zero moments of the first and third orders

$$M[V_{r_1} V_{r_2} V_{r_3}] = 0 \quad (r_1,\ r_2,\ r_3 = 1,\ 2,\ \ldots,\ m). \qquad (IV.59)$$

In the case in question to the system of equations (IV.45)-(IV.48) will be supplemented other equations of the form

$$\sum_{s=1}^{N} \alpha_s \xi_{r_1 s} \xi_{r_2 s} \xi_{r_3 s} = 0 \quad (r_1,\ r_2,\ r_3 = 1,\ 2,\ \ldots,\ m). \qquad (IV.60)$$

According to formula (IV.11) a number of solutions of system of equations G, with which can be guaranteed the existence of real values $\alpha_s$ and $\xi_{r_k s}$, determined by equations (IV.45)-(IV.48) and (IV.60), in this case is equal

$$N = C^3_{m+3} = \frac{(m+3)(m+2)(m+1)}{1 \cdot 2 \cdot 3}.$$

Page 113.

This number can be very considerable. Thus, already with m=5 we obtain N=56. Meanwhile it is not difficult to select values $\xi_{r_\kappa}$, so as to substantially reduce the number of necessary solutions of system G. Let us select values $\xi_{r_\kappa}$, in accordance with Tables IV.2.

Then, obviously, we will have relationships/ratios

$$N = 2m, \qquad \text{(IV.61)}$$
$$\xi_{r_1 s}\xi_{r_2 s} = 0 \text{ при } (r_1 \neq r_2); \qquad \text{(IV.62)}$$
$$\xi_{r_1 s}\xi_{r_2 s} = \xi_r^2 \text{ при } (r_1 = r_2 = r). \qquad \text{(IV.63)}$$

Key: (1). with.

On the basis of equality (IV.62) we consist that all equations (IV.47) during this selection of values $\xi_{r_\kappa s}$ are satisfied; however, as far as equations (IV.46), (IV.48) and (IV.60) are concerned, they are reduced to the form

$$\alpha_{2r-1} - \alpha_{2r} = 0 \quad (r = 1, 2, \ldots, m); \qquad \text{(IV.64)}$$
$$(\alpha_{2r-1} + \alpha_{2r})\xi_r^2 = \sigma_r^2. \qquad \text{(IV.65)}$$

Choosing everything $\alpha_s$ identical ($\alpha_s = \alpha$), we satisfy equations

(IV.64), and from equations (IV.45) and (IV.65) we obtain

$$\alpha_s = \alpha = \frac{1}{2m}; \qquad \text{(IV.66)}$$

$$\xi_r = \sigma_r \sqrt{m} \quad (r = 1, 2, \ldots, m). \qquad \text{(IV.67)}$$

If we substitute into the system of equations G for random parameters $V_r$ equivalent disturbances/perturbations $\xi_r$ and $-\xi_r$ and to solve this system, then we will obtain 2m the values of the solutions of the form

$$\left.\begin{aligned} y_r &= \varphi(t, 0, \ldots, 0, \xi_r, 0, \ldots, 0); \\ y_r^{*} &= \varphi(t, 0, \ldots, 0, -\xi_r, 0, \ldots, 0). \end{aligned}\right\} \qquad \text{(IV.68)}$$

Table IV.2.

| $s$ | $r$: 1 | 2 | 3 | ............... | $m-1$ | $m$ |
|---|---|---|---|---|---|---|
| 1 | $\xi_1$ | 0 | 0 | ............ | 0 | 0 |
| 2 | $-\xi_1$ | 0 | 0 | ............ | 0 | 0 |
| 3 | 0 | $\xi_2$ | 0 | ............ | 0 | 0 |
| 4 | 0 | $-\xi_2$ | 0 | ............ | 0 | 0 |
| ..... | ..... | ..... | ..... | ..... | ..... | ..... |
| $2m-1$ | 0 | 0 | 0 | ............ | 0 | $\xi_m$ |
| $2m$ | 0 | 0 | 0 | ............ | 0 | $-\xi_m$ |

Page 114.

The mathematical expectation M[Y] on the basis of equality (IV.10) takes the form

$$M[Y]=\frac{\sum_{r=1}^{m}(y_r+y_r^*)}{2m}, \qquad \text{(IV.69)}$$

and the dispersion of value Y can be determined according to the formula

$$D[Y]=M[Y^2]-(M[Y])^2=\frac{\sum_{r=1}^{m}(y_r^2+y_r^{*2})}{2m}-(M[Y])^2. \quad \text{(IV.70)}$$

This formula easily is reduced to the form, which can prove to be more convenient during the practical calculations, namely:

$$D[Y]=M[Y-M[Y]]^2=\frac{1}{2m}\left[\sum_{r=1}^{m}(y_r-M[Y])^2+(y_r^*-M[Y])^2\right]. \qquad \text{(IV.71)}$$

Formulas (IV.70) and (IV.71) especially are simplified, if in the system of equations G single parameter $V_r = V$ is contained. Actually/really, under these conditions

$$M[Y] = \frac{y+y^*}{2}; \qquad (IV.72)$$

$$D[Y] = \left(\frac{y-y^*}{2}\right)^2. \qquad (IV.73)$$

i.e. mathematical expectation is equal to the average of the solutions of system of equations G, which are obtained with the substitution instead of parameter V of values $\sigma_V$ and $-\sigma_V$, and dispersion is equal to the square of a difference in these solutions.

With the high values of m can seem that values $\xi_r$ will exceed maximally possible values of parameters $V_r$. In this case the error of expression (IV.3), caused by the rejection of the remainder of resolution, can prove to be inadmissibly to large. However, it is possible to easily obtain formulas for determining the mathematical expectation of value Y, which do not require the substitution of high values $\xi_r$. In fact, let us select $\xi_r$, that figure in by tables IV.2, proportional to the appropriate standard deviations $\sigma_r$:

$$\xi_r = \lambda\sigma_r \quad (r = 1, 2, \ldots, m), \qquad (IV.74)$$

where λ - the constant coefficient, not depending on r.

Page 115.

It is possible to obtain for the mathematical expectation of function Y following formula [34]:

$$M[Y] = \frac{1}{2\lambda^2} \sum_{r=1}^{m} (y_r + y_r^*) + y_0\left(1 - \frac{m}{\lambda^2}\right), \qquad \text{(IV.75)}$$

where $y_r$ and $y_r^*$ are determined by equalities (IV.68);

$$y_0 = \varphi(t, 0, \ldots, 0);$$

with $\lambda=\sqrt{m}$ this formula coincides with formula (IV.69).

If all values $V_r$ are subordinated to the normal distribution law, then they usually assume that their limiting values virtually do not exceed $3\sigma_r$. After selecting $\lambda=3$, from formula (IV.75) we will obtain

$$M[Y] = \frac{1}{18} \sum_{r=1}^{m} (y_r + y_r^*) + y_0\left(1 - \frac{m}{9}\right). \qquad \text{(IV.76)}$$

5. Determination of the probabilistic characteristics of the output coordinate of nonlinear system for case of q=5.

Upon transfer in formula (IV.3) to the polynomials of the fifth order relatively $V_r$ the complexity of the application of the method of equivalent disturbances/perturbations sharply grows. In work [34] are obtained formulas for case of q=5, requiring N solutions of equations G, which describe the dynamics of nonlinear automatic control system, where

$$N = 2m^2 + 1. \qquad \text{(IV.77)}$$

These formulas were based on the exact account in expansion (IV.3) of all values to the fifth order inclusively. Meanwhile if one assumes that the combined simultaneous effect of four different parameters $V_r$ on the output coordinate Y of system can be disregarded/neglected, i.e., if we suppose that the partial derivatives of the form

$$\left(\frac{\partial^4\varphi}{\partial V_{r_1}\partial V_{r_2}\partial V_{r_3}\partial V_{r_4}}\right)_0=0,$$

where $r_1$, $r_2$, $r_3$, $r_4$, is not pair-wise equal to each other, then instead of the complicated formulas, given in work [34], it is possible to obtain considerably more economical expressions. Let us consider this version of the method of equivalent disturbances/perturbations.

Let random variables $V_r (r=1, 2, \ldots, m)$ be independent, subordinated to one and the same distribution law and have the moments/torques

$M[V_r]=0$; $M[V_r^2]=\sigma^2$; $M[V_r^3]=\mu_3$; $M[V_r^4]=\mu_4$ and so forth.

Page 116.

Then passing both parts of equality (IV.3) of the random

variables to their mathematical expectations, let us find

$$M[Y] = a_y = \varphi_0 + 0 + \left[\frac{1}{2!}\sum_{r=1}^{m}\left(\frac{\partial^2\varphi}{\partial V_r^2}\right)_0\right]\sigma^2 + \\ + \left[\frac{1}{3!}\sum_{r=1}^{m}\left(\frac{\partial^3\varphi}{\partial V_r^3}\right)_0\right]\mu_3 + \left[\frac{1}{4!}\sum_{r=1}^{m}\left(\frac{\partial^4\varphi}{\partial V_r^4}\right)_0\right]\mu_4 + \\ + \left[\frac{1}{4!}\sum_{r_1=1}^{m}\sum_{r_2=1}^{m}\left(\frac{\partial^4\varphi}{\partial V_{r_1}^2 \partial V_{r_2}^2}\right)_0\right]\sigma^4 + \ldots \qquad \text{(IV.78)}$$

If we assume the allocation of values $V_r$ symmetrical, then the condition

$$M\left[V_r^{2\kappa+1}\right] = 0 \quad (\kappa = 1, 2, \ldots)(r = 1, 2, \ldots, m)$$

will be satisfied.

In this case equality (IV.78) will take the following form:

$$a_y = \varphi_0 + \left[\frac{1}{2!}\sum_{r=1}^{m}\left(\frac{\partial^2\varphi}{\partial V_r^2}\right)_0\right]\sigma^2 + \left[\frac{1}{4!}\sum_{r=1}^{m}\left(\frac{\partial^4\varphi}{\partial V_r^4}\right)_0\right]\mu_4 + \\ + \left[\frac{1}{4!}\sum_{r_1=1}^{m}\sum_{r_2=1}^{m}\left(\frac{\partial^4\varphi}{\partial V_{r_1}^2 \partial V_{r_2}^2}\right)_0\right]\sigma^4. \qquad \text{(IV.79)}$$

Task in selecting of equivalent disturbances/perturbations $\xi_{1s}, \xi_{2s}, \ldots, \xi_{ms} (s = 1, 2, \ldots, N)$, during input/introduction of which it is possible to sufficiently economically carry out a computation of values $a_y$.

Let us introduce the following nomenclature:

215



$$\left.\begin{aligned}
&\varphi_0 = S_0;\\
&\frac{1}{1!}\sum_{r=1}^{m}\left(\frac{\partial\varphi}{\partial V_r}\right)_0 = S_1;\\
&\frac{1}{2!}\sum_{r=1}^{m}\left(\frac{\partial^2\varphi}{\partial V_r^2}\right)_0 = S_2;\\
&\frac{1}{2!}\sum_{r_1=1}^{m}\sum_{r_2=1}^{m}\left(\frac{\partial^2\varphi}{\partial V_{r_1}\partial V_{r_2}}\right)_0 = S_{11}\quad (r_1\neq r_2);\\
&\frac{1}{3!}\sum_{r=1}^{m}\left(\frac{\partial^3\varphi}{\partial V_r^3}\right)_0 = S_3;\\
&\frac{3}{3!}\sum_{r_1=1}^{m}\sum_{r_2=1}^{m}\left(\frac{\partial^3\varphi}{\partial V_{r_1}^2\partial V_{r_2}}\right)_0 = S_{21}\quad (r_1\neq r_2);\\
&\frac{1}{3!}\sum_{r_1=1}^{m}\sum_{r_2=1}^{m}\sum_{r_3=1}^{m}\left(\frac{\partial^3\varphi}{\partial V_{r_1}\partial V_{r_2}\partial V_{r_3}}\right)_0 = S_{111}\\
&(r_1\neq r_2;\quad r_1\neq r_3;\quad r_2\neq r_3)\ \text{and so forth.}
\end{aligned}\right\}\qquad \text{(IV.80)}$$

Page 117.

Then expression (IV.78) is converted to the form

$$\varphi_y = S_0 + S_2\sigma^2 + S_4\mu_4 + S_{22}\sigma^4. \qquad \text{(IV.81)}$$

For obtaining the sums of $S_0$, $S_2$, $S_4$, $S_{22}$, that figure in this equality, it is possible to use input/introduction into the initial equations of the system of values $\xi_{rs}$ being investigated instead of random parameters $V_r$. In this case the following version of the selection of values $\xi_{rs}$ is feasible

First,

$$\xi_{rs} = 0 \quad (r = 1, 2, \ldots, m),$$

then on the basis of expansion (IV.3) and equalities (IV.80) let us

find

$$Y = y_0 = \varphi_0 = S_0. \qquad (IV.82)$$

In the second place, let us select values $\xi_{rs}$ according to the equality

$$\xi_{rs} = \xi_s \quad \begin{pmatrix} r = 1, 2, \ldots, m \\ s = 1, 2, \ldots, u \end{pmatrix};$$

in which $\xi_s$ and they will be determined by u below. Taking into account adopted designations (IV.80) series/row (IV.3) can be represented in the form

$$\begin{aligned} y = y_s = S_0 + S_1\xi_s + (S_2 + S_{11})\xi_s^2 + (S_3 + S_{21} + S_{111})\xi_s^3 + \\ + (S_4 + S_{31} + S_{22} + S_{211} + S_{1111})\xi_s^4 + \ldots \end{aligned} \qquad (IV.83)$$

Thirdly, let us select values $\xi_{rs}$ in accordance with Tables IV.3. Such tables are written/recorded for j=1, by 2, ..., v. Values $v$, $\theta_j$, $\xi_j$ will be also determined subsequently. If we fulfill the substitution of values $\xi_{rs}$ into equality (IV.3) in accordance with Tables IV.3 and values of coordinate y obtained in this case to sum up, then we will obtain

$$\left.\begin{aligned} \sum_{s=1}^{m} y_{sj} = S_0 m + S_1[(m-1)\xi_j + \theta_j] + S_2[(m-1)\xi_j^2 + \theta_j^2] + \\ + S_{11}[(m-2)\xi_j^2 + 2\xi_j\theta_j] + S_3[(m-1)\xi_j^3 + \theta_j^3] + \\ + S_{21}[(m-2)\xi_j^3 + \xi_j^2\theta_j + \xi_j\theta_j^2] + S_{111}[(m-3)\xi_j^3 + \\ + 3\xi_j^2\theta_j] + S_4[(m-1)\xi_j^4 + \theta_j^4] + S_{31}[(m-2)\xi_j^4 + \\ + \xi_j^3\theta_j + \xi_j\theta_j^3] + S_{22}[(m-2)\xi_j^4 + 2\xi_j^2\theta_j^2] + \\ + S_{211}[(m-3)\xi_j^4 + 2\xi_j^3\theta_j + \xi_j^2\theta_j^2] + S_{1111}[(m-4)\xi_j^4 + \\ + 4\xi_j^3\theta_j] + \ldots \quad (j = 1, 2, \ldots, v) \end{aligned}\right\} \qquad (IV.84)$$

Unknown values $S_{r_1 r_2 \ldots r_k}$ in the principle can be found from linear equations (IV.83), (IV.84) taking into account (IV.82). However, unfortunately, between the coefficients, of the unknowns $S_{22}$, $S_{211}$ and $S_{1111}$, there is a linear dependence.

Page 118.

Therefore without the further conditions, which connect these unknowns, it is impossible to solve task. Let us consider the case now one important for the practice, when q=5, $S_{1111}=0$ (by hypothesis, formulated above).

From equality (IV.84) it is easy to establish that, choosing

$$\xi_j = -\theta_j = -\theta,$$

we will obtain

$$\begin{aligned}\Sigma_1 = S_0 m - S_1 (m-2)\theta + S_2 m\theta^2 + S_{11}(m-4)\theta^2 - S_3(m-2)\theta^3 - \\ - S_{21}(m-2)\theta^3 - S_{111}(m-6)\theta^3 + S_4 m\theta^4 + S_{31}(m-4)\theta^4 + \\ + S_{22} m\theta^4 + S_{211}(m-4)\theta^4 + S_V,\end{aligned}$$

where by symbol $S_V$ they are designated the members of the fifth order of smallness relatively $\theta$. Let us fulfill now the substitution of values $\xi_j$ and $\theta_j$, opposite on the sign, i.e., let us take

$$\xi_j = -\theta_j = \theta.$$

Then equality (IV.84) is reduced to the form

$$\Sigma_{II} = S_0 m + S_1 (m-2)\theta + S_2 m\theta^2 + S_{11}(m-4)\theta^2 + S_3 (m-2)\theta^3 + S_{21}(m-2)\theta^3 + S_{111}(m-6)\theta^3 + S_4 m\theta^4 + S_{31}(m-4)\theta^4 + S_{22} m\theta^4 + S_{211}(m-4)\theta^4 - S_V.$$

Calculating average $\frac{\Sigma_I + \Sigma_{II}}{2}$, let us find

$$\frac{\Sigma_I + \Sigma_{II}}{2} = S_0 m + S_2 m\theta^2 + S_{11}(m-4)\theta^2 + S_4 m\theta^4 + (S_{31} + S_{211})(m-4)\theta^4 + S_{22} m\theta^4. \quad \text{(IV.85)}$$

Let us note that for computing this average it is necessary to fulfill 2m the integrations of the reference system of equations.

Table IV.3.

| $s$ | $r$ | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | ........ | $m-1$ | $m$ |
| 1 | $\theta_j$ | $\xi_j$ | ...... | $\xi_j$ | $\xi_j$ |
| 2 | $\xi_j$ | $\theta_j$ | ...... | $\xi_j$ | $\xi_j$ |
| ...... | ...... | ...... | ...... | ...... | ...... |
| $m-1$ | $\xi_j$ | $\xi_j$ | ...... | $\theta_j$ | $\xi_j$ |
| $m$ | $\xi_j$ | $\xi_j$ | ...... | $\xi_j$ | $\theta_j$ |

Page 119.

It is analogous on the basis of equality (IV.83) when $\xi_s = \xi$ and $\xi_s = -\xi$ we will obtain

$$\begin{aligned}\Sigma_{III} &= S_0 + S_1\xi + (S_2 + S_{11})\xi^2 + (S_3 + S_{21} + S_{111})\xi^3 + \\ &\quad + (S_4 + S_{31} + S_{22} + S_{211})\xi^4 + S_V';\\ \Sigma_{IV} &= S_0 - S_1\xi + (S_2 + S_{11})\xi^2 - (S_3 + S_{21} + S_{111})\xi^3 + \\ &\quad + (S_4 + S_{31} + S_{22} + S_{211})\xi^4 - S_V'.\end{aligned}$$

During the recording of these equalities sums $S_{1111}$ are rejected/thrown by hypothesis, and symbol $S_V'$ designated the members of the fifth order of smallness relatively $\xi$. Hence we will obtain

$$\frac{\Sigma_{III} + \Sigma_{IV}}{2} = S_0 + S_2\xi^2 + S_{11}\xi^2 + S_4\xi^4 + (S_{31} + S_{211})\xi^4 + S_{22}\xi^4. \tag{IV.86}$$

For computing this average two integrations of the reference system of equations are required. In equalities (IV.85) and (IV.86) "excess" components/terms/addends are those, into which the variables

$S_{11}$, $S_{31}$, $S_{211}$ enter, which escapes/ensues from the examination of formula (IV.81). Summarizing piecemeal equalities (IV.85) and (IV.86) with some coefficients $\alpha_1$ and $\alpha_2$, we will be freed from "excess" components/terms/addends. For this, obviously, must be satisfied the conditions

$$\alpha_1(m-4)\theta^2+\alpha_2\xi^2=0;$$
$$\alpha_1(m-4)\theta^4+\alpha_2\xi^4=0.$$

This system has solutions, different from the zero ones under the condition

$$\theta=\pm\xi. \qquad (IV.87)$$

Then

$$\alpha_2=-(m-4)\alpha_1$$

and

$$\sum=\frac{\alpha_1}{2}(\Sigma_{I}+\Sigma_{II})+\frac{\alpha_2}{2}(\Sigma_{III}+\Sigma_{IV})=S_0(\alpha_1 m+\alpha_2)+S_2(\alpha_1 m+\alpha_2)\theta^2+$$
$$+S_4(\alpha_1 m+\alpha_2)\theta^4+S_{22}(\alpha_1 m+\alpha_2)\theta^4=4\alpha_1(S_0+S_2\theta^2+S_4\theta^4+S_{22}\theta^4). \qquad (IV.88)$$

The obtained expression has a structure of equality (IV.81); however, coefficients with sums of $S_0$, $S_2$, $S_4$ and $S_{22}$ in these expressions different. Therefore expression (IV.88) cannot be used directly for determining the mathematical expectation $a_\nu$.

Page 120.

Let us select according to Tables IV.3 substitution of values $\theta_j$ and $\xi_j$ as follows:

$$\text{a) } \theta_j = \psi;\quad \xi_j = 0;$$
$$\text{b) } \theta_j = -\psi;\quad \xi_j = 0.$$

Then from formula (IV.84) let us find two sums

$$\Sigma_{V} = S_0 m + S_1\psi + S_2\psi^2 + S_3\psi^3 + S_4\psi^4 + S_5\psi^5;$$
$$\Sigma_{VI} = S_0 m - S_1\psi + S_2\psi^2 - S_3\psi^3 + S_4\psi^4 - S_5\psi^5;$$

hence

$$\frac{\Sigma_V + \Sigma_{VI}}{2} = S_0 m + S_2\psi^2 + S_4\psi^4. \qquad \text{(IV.89)}$$

For computing this average it is necessary to fulfill 2m the integrations of the reference system of equations. After multiplying equality (IV.89) piecemeal to certain coefficient $\beta$ and prevailing with equality (IV.88), let us lead obtained thus expression to the form

$$\frac{\alpha_1}{2}(\Sigma_I + \Sigma_{II}) + \frac{\alpha_2}{2}(\Sigma_{III} + \Sigma_{IV}) + \frac{\beta}{2}(\Sigma_V + \Sigma_{VI}) =$$
$$= S_0 + S_2\sigma^2 + S_4\mu_4 + S_{22}\sigma^4. \qquad \text{(IV.90)}$$

For this of value $\alpha_1$, $\beta$, $\theta$, $\psi$ must satisfy the equations

$$4\alpha_1 + m\beta = 1;$$
$$4\alpha_1\theta^2 + \beta\psi^2 = \sigma^2;$$
$$4\alpha_1\theta^4 + \beta\psi^4 = \mu_4;$$
$$4\alpha_1\theta^4 = \sigma_4.$$

For the normal random number distribution $V$, the equality

$$\mu_4 = 3\sigma^4$$

will occur.

Solving under this condition the obtained system of equations, let us find

$$\left.\begin{aligned} \alpha_1 &= \frac{1}{4}\left(\frac{m-2}{m+2}\right)^2; \\ \beta &= \frac{8}{(m+2)^2}; \\ \theta &= \sigma\sqrt{\frac{m+2}{m-2}}; \\ \psi &= \sigma\sqrt{\frac{m+2}{2}}. \end{aligned}\right\} \qquad (IV.91)$$

Page 121.

Thus, taking into account equalities (IV.81)-(IV.90), let us find

$$\sigma_y = \frac{1}{2}\left[\alpha_1(\Sigma_I + \Sigma_{II}) + \alpha_2(\Sigma_{III} + \Sigma_{IV}) + \beta(\Sigma_V + \Sigma_{VI})\right], \quad (IV.92)$$

where the coefficients $\alpha_1$, $\alpha_2$, $\beta$ are calculated from formulas (IV.91). Sums $\Sigma_I - \Sigma_{VI}$ are determined as a result of the following substitutions of values $\theta$ and $\psi$:

а) для $\Sigma_{I}$ (1)

| $s$ | $V_1$ | $V_2$ | $\cdots$ | $V_{m-1}$ | $V_m$ |
|---|---|---|---|---|---|
| 1 | $+\theta$ | $-\theta$ | $\cdots$ | $-\theta$ | $-\theta$ |
| 2 | $-\theta$ | $+\theta$ | $\cdots$ | $-\theta$ | $-\theta$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $m-1$ | $-\theta$ | $-\theta$ | $\cdots$ | $+\theta$ | $-\theta$ |
| $m$ | $-\theta$ | $-\theta$ | $\cdots$ | $-\theta$ | $+\theta$ |

b) для (1) $\Sigma_{II}$

| $s$ | $V_1$ | $V_2$ | $\cdots$ | $V_{m-1}$ | $V_m$ |
|---|---|---|---|---|---|
| 1 | $-\theta$ | $+\theta$ | $\cdots$ | $+\theta$ | $+\theta$ |
| 2 | $+\theta$ | $-\theta$ | $\cdots$ | $+\theta$ | $+\theta$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $m-1$ | $+\theta$ | $+\theta$ | $\cdots$ | $-\theta$ | $+\theta$ |
| $m$ | $+\theta$ | $+\theta$ | $\cdots$ | $+\theta$ | $-\theta$ |

c) для (1) $\Sigma_{III}$

| $s$ | $V_1$ | $V_2$ | $\cdots$ | $V_{m-1}$ | $V_m$ |
|---|---|---|---|---|---|
| 1 | $+\theta$ | $+\theta$ | $\cdots$ | $+\theta$ | $+\theta$ |

d) для (1) $\Sigma_{IV}$

| $s$ | $V_1$ | $V_2$ | $\cdots$ | $V_{m-1}$ | $V_m$ |
|---|---|---|---|---|---|
| 1 | $-\theta$ | $-\theta$ | $\cdots$ | $-\theta$ | $-\theta$ |

e) для (1) $\Sigma_{V}$

| $s$ | $V_1$ | $V_2$ | $\cdots$ | $V_{m-1}$ | $V_m$ |
|---|---|---|---|---|---|
| 1 | $\psi$ | 0 | $\cdots$ | 0 | 0 |
| 2 | 0 | $\psi$ | $\cdots$ | 0 | 0 |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $m-1$ | 0 | 0 | $\cdots$ | $\psi$ | 0 |
| $m$ | 0 | 0 | $\cdots$ | 0 | $\psi$ |

f) для (1) $\Sigma_{VI}$

| $s$ | $V_1$ | $V_2$ | $\cdots$ | $V_{m-1}$ | $V_m$ |
|---|---|---|---|---|---|
| 1 | $-\psi$ | 0 | $\cdots$ | 0 | 0 |
| 2 | 0 | $-\psi$ | $\cdots$ | 0 | 0 |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $m-1$ | 0 | 0 | $\cdots$ | $-\psi$ | 0 |
| $m$ | 0 | 0 | $\cdots$ | 0 | $-\psi$ |

Key: (1). for.

Page 122.

Values $\theta$ and $\psi$ are calculated from formulas (IV.91). It is obvious that for the computations for formula (IV.92) it is necessary to fulfill in all

$$N = 4m + 2 = 2(2m + 1) \qquad \text{(IV.93)}$$

of the integrations of the initial equations of the automatic control system being investigated. This number is substantially less than determined from formulas (IV.77). For the comparison let us give the results of the calculations of number N according to these formulas:

| | (1) Формула | |
|---|---|---|
| $m$ | (IV.77) | (IV.93) |
| 5 | 51 | 22 |
| 10 | 201 | 42 |
| 15 | 451 | 62 |
| 20 | 801 | 82 |
| 30 | 1801 | 122 |

Key: (1). Formula.

Given data testify about the large gain in the space of the necessary computations, obtained during the use of formula (IV.92). However, in this case it is necessary to keep in mind that this formula is obtained under the assumption of the smallness of sum $S_{1111}$.

Example. Let

$Y = V_1^2 \sin^2 V_2 + V_3^2 \sin^2 V_4$, i.e. m=4.

Let us determine $a_Y$, assuming that values $V_1$, $V_2$, $V_3$, $V_4$ are independent and subordinated to the normal distribution law with the parameters

$$a_{V_r} = 0;\ \sigma_{V_r} = \sigma\ (r = 1, 2, 3, 4).$$

Through formulas (IV.91) we find

$$\theta = \sigma\sqrt{3};\quad \psi = \sigma\sqrt{3}.$$

Further, substituting in the expression for Y value $\theta$ and $\psi$ in accordance with points/items a) - f), let us find

$$\Sigma_{I} = \Sigma_{II} = 24\sigma^2 \sin^2 \sigma\sqrt{3};$$
$$\Sigma_{III} = \Sigma_{IV} = 6\sigma^2 \sin^2 \sigma\sqrt{3};$$
$$\Sigma_{V} = \Sigma_{VI} = 0.$$

According to formulas (IV.91) and (IV.92) let us determine

$$\alpha_1 = \frac{1}{36};\ \alpha_2 = 0;\ \beta = \frac{2}{9};$$
$$a_Y = \frac{2}{3}\sigma^2 \sin^2 \sigma\sqrt{3}.$$

Page 123.

Task allows/assumes exact solution. After comparatively simple transformations we can find

$$a_Y = 1 - e^{-2\sigma^2}.$$

Given below the comparison of exact $a_Y$ and approximate value $\tilde{a}_Y$ for $\sigma=0$; $\sigma=\frac{1}{2\sqrt{3}}$ and $\sigma=\frac{1}{\sqrt{3}}$:

| $\sigma$ | $0$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{\sqrt{3}}$ |
|---|---|---|---|
| $a_Y$ | 0 | 0,153 | 0,472 |
| $\tilde{a}_Y$ | 0 | 0,154 | 0,483 |

Given data speak about the satisfactory coincidence of results.

Page 124.

Chapter V.

# INTERPOLATION METHOD OF ANALYZING THE ACCURACY OF NONLINEAR AUTOMATIC CONTROL SYSTEMS.

## 1. Generalized formulation of task of analysis of dynamics of automatic control systems taking into account of random disturbances and parameters.

As it was indicated in Chapter I, the solution of the problems of determining the statistical characteristics of nonlinear automatic systems with the help of the analytical methods is possible only in the simplest cases. Therefore at present considerable development and use/application obtained the approximate numerical methods of the solution of similar problems. Partially these methods were reflected in the previous chapters.

In present chapter given certain generalization of the formulation of the problem of the analysis of the dynamics of

automatic control systems is proposed one of the possible methods of its solution.

Page 125.

Let as before the dynamics of nonlinear automatic control system be described by equations (VI.1) of the form

$$\frac{dY_i}{dt} = f_i(Y_1, Y_2, \ldots, Y_n, V_1, V_2, \ldots, V_m, t) \qquad (V.1)$$
$$(i = 1, 2, \ldots, n),$$

where $Y_i$ — output coordinates of system;

$V_j$ — random parameters.

Without decreasing generality, it can be assumed that the initial conditions during the solution of equations are zero.

Let us substitute the now following task.

Are preset the probabilistic characteristics of random parameters $V_1, V_2, \ldots, V_m$ in the form of their moments/torques or distribution laws and is selected the system of functionals $\Phi_i$ from the output coordinates:

$$\Phi_i(t, Y_1, Y_2, \ldots, Y_n), \quad i = 1, 2, \ldots \qquad (V.2)$$

Furthermore, are preset some functions

$$\chi_\kappa \cdot \chi_\kappa(\Phi_l), \quad \kappa - 1, 2, \ldots \qquad (V.3)$$

It is necessary along preset system (V.1) and according to preset probabilistic characteristics of random parameters $V_1, V_2, \ldots, V_m$ to determine the mathematical expectations

$$\upsilon_\kappa = M[\chi_\kappa(\Phi_l)], \quad \kappa = 1, 2, \ldots \qquad (V.4)$$

In the particular case of value $M[\chi_\kappa(\Phi_l)]$ can be mathematical expectations and dispersions of the output coordinates of nonlinear system, mathematical expectations and dispersions of the preset functions of these coordinates, etc.

In the previous chapter the version of setting the task, when the unknown values are mathematical expectations, dispersions and moments/torques of connection/communication for the output coordinates of nonlinear system, was examined.

2. Algorithms of the numerical methods of determining the statistical

characteristics of automatic systems.

The simplest computing circuit for determining the values $M[\chi_n(\Phi_i)]$ is obtained during the use/application of a method for statistical testing (Monte Carlo method).

It is similar to the simplest versions of task, examined in chapter I, the sequence of calculations according to the method for statistical testing in this case consists of the following stages:

1. The sample of random numbers $V_1, V_2, \ldots, V_m$, entering the right sides of system (V.1) is determined. The sample of the values of values is constructed in accordance with the preset laws of distribution of the probabilities of random variables $V_1, V_2, \ldots, V_m$. For fulfilling this operation at present proposed many different methods [15] are standard subroutines for different classes of ETsVM [digital computer].

2. With the help of numerical methods integration of system of differential equations (V.1) is realized under zero initial conditions and at concrete/specific/actual numerical values of random variables $V_1, V_2, \ldots, V_m$, of those obtained in point/item 1.

Page 126.

3. Points/items 1 and 2 are repeated repeatedly. As a result several versions of the solutions of the system

$$y_{1\kappa}(t),\ y_{2\kappa}(t),\ \ldots,\ y_{n\kappa}(t),\quad \kappa=1,\ 2,\ \ldots,\ N, \qquad \text{(V.5)}$$

where k - number of version, are obtained.

Approximate values of the required statistical characteristics are designed from the formula

$$M[\chi(\Phi)]\approx\frac{1}{N}\sum_{\kappa=1}^{N}\chi[\Phi(t,\ y_{1\kappa},\ y_{2\kappa},\ \ldots,\ y_{n\kappa})]. \qquad \text{(V.6)}$$

With an increase in the number of versions N the error decreases with a speed of $\frac{1}{\sqrt{N}}$, therefore for obtaining the acceptable accuracy of result by the method for statistical testing it is necessary to realize numerical integration of system (V.1) for a large number of versions, values of values $V_1,\ V_2,\ldots,\ V_m$, which in the majority of the cases leads to the excessively high expenditures of the operating time of computer(s). Therefore more economical methods are proposed.

For computing of mathematical expectations and dispersions of output coordinates and moments/torques of higher orders can be used the method of equivalent disturbances/perturbations, presented in the previous chapter and which uses a representation of output

coordinates in the form of the functions of the time and random parameters $V_1, V_2, \ldots, V_m$, i.e. in the form

$$Y_i = y_i(t, V_1, V_2, \ldots, V_m). \qquad (V.7)$$

The calculation formulas of this method outwardly differ from the formulas of the method for statistical testing only by the presence of multipliers $\alpha_\kappa$. Using the adopted in this chapter designations and a mathematical formulation of the task in question, it is possible these formulas to represent in the following form:

$$M[\chi(\Phi)] = \sum_{\kappa=1}^{N} \alpha_\kappa \chi[\Phi(t, y_{1\kappa}, y_{2\kappa}, \ldots, y_{n\kappa})]. \qquad (V.8)$$

During the use of a method of equivalent disturbances/perturbations there is no need for realizing a procedure of obtaining random numbers. This procedure, as it was shown in chapter IV, was replaced by the solution of certain auxiliary nonlinear system of algebraic equations relative to coefficients $\alpha_\kappa$ and versions of the values of random variables $V_1, V_2, \ldots, V_m$, with which necessary to make numerical integration.

Page 127.

Unfortunately, method is worked out only for calculating the

moment characteristics, while the method for statistical testing in the principle it is possible to apply, also, for determining the laws of distribution of the output coordinates of systems.

In contrast to the method of equivalent disturbances/perturbations the interpolation method, proposed in work [109], is worked out also for the calculation of the probability distribution function [108] at the output of nonlinear systems, i.e., it possesses the same universality, as the method for statistical testing.

The calculation formulas of interpolation method can be in the general case represented in the form

$$M[\chi(\Phi)] = \sum_{\kappa=1}^{N} \rho_\kappa \chi[\Phi(t, y_{1\kappa}, y_{2\kappa}, \ldots, y_{n\kappa})], \qquad \text{(V.9)}$$

where $\rho_\kappa$ — some constant numbers.

When the versions of values of random variables are selected one and the same for the interpolation method and for the method of equivalent disturbances/perturbations, and number $\rho_\kappa = \alpha_\kappa$, that the method of equivalent disturbances/perturbations coincides with the interpolation identically.

However, due to the optimum selection of the nodes of interpolation with one and the same number of versions of the solutions of system of equations (V.1) interpolation method gives higher accuracy than the method of equivalent disturbances/perturbations. Therefore it is expedient to apply interpolation method for the optimum nodes of interpolation.

In appendix 6 are placed the tables of optimum numbers (Christoffel number) and the optimum versions of the nodes of interpolation (Chebyshev's nodes) for the random variables, which have the uniform (table 1) and normal or exponential (table 2) distribution laws.

During the practical calculations it is necessary to convert initial random variables to one or the other standard form in the dependence on the preset distribution laws.

If random variable $V_j$ has the uniform distribution law in gap/interval $[a_j, b_j]$, that nodes of Chebyshev's type are designed from the formula

$$V_{l\kappa_j} \quad \frac{b_j - a_j}{2}\lambda_{\kappa q_j} + \frac{b_j + a_j}{2}, \qquad (V.10)$$

where $\lambda_{\kappa q_j}$ — standard nodes of Chebyshev for the uniform distribution

law in gap/interval [1.1].

If random variable $V_j$ has the normal law of distribution of probability with parameters $a_j$ and $\sigma_j^2$ (mathematical expectation and dispersion), then Chebyshev's nodes are designed from the formula

$$V_{l\kappa_j} = a_j + \sigma_j\lambda_{\kappa q_j}. \qquad \text{(V.11)}$$

Standard allocation has parameters $\bar{V}=0$; $\sigma=1$.

Page 128.

If random variable $V_j$ has the exponential law of distribution of probability with the mathematical expectation, equal to $\frac{1}{\mu_j}$, then Chebyshev's nodes are determined from the formula

$$V_{l\kappa_j} = \mu_j\lambda_{\kappa q_j}. \qquad \text{(V.12)}$$

Standard allocation has a parameter $\mu=1$. Christoffel numbers need not be recalculated.

3. Calculation formulas of interpolation method.

After the reduction of random variables to the standard form

(under the condition of their independence) the calculation formula of interpolation method can be recorded in the following form:

$$M[\chi]\approx \sum_{\kappa_1\kappa_2\ldots\kappa_m} \chi(t, V_{1\kappa_1}, V_{2\kappa_2}, \ldots, V_{m\kappa_m}) \prod_{j=1}^{m} \rho_{\kappa_j}, \qquad (V.13)$$

where the addition is realized on all indices $\kappa_1=1, 2, \ldots,$

$$q_1;\ \kappa_2=1, 2, \ldots, q_2;\ \ldots;\ k_m=1, 2, \ldots, q_m;$$

$q_1$ - number of the different values of the random variable $V_1$, utilized during calculations;

$q_2$ - number of the different values of the random variable $V_2$, utilized during calculations and so forth;

$V_{1\kappa_1}, V_{2\kappa_2}, \ldots, V_{m\kappa_m}$ — the diverse variants of values, obtained from Tables 1 and 2 appendices 6 by path led of initial random variables to the standard form;

$\rho_{\kappa_j}$ — Christoffel number for standard random variable $\lambda_j$.

Formula (V.13) can be defined concretely, if to consider the specific forms of the assignment to characteristic function $\chi$.

For example, during the calculation of the mathematical expectations of output coordinates formula (V.13) takes the following form:

$$\overline{Y}_l \quad M[Y_l(t)] \approx \sum_{\kappa_1\kappa_2\ldots\kappa_m} y_l(t, V_{1\kappa_1}, V_{2\kappa_2}, \ldots, V_{m\kappa_m}) \prod_{j=1}^{m} \rho_{\kappa_j}, \quad (V.14)$$

$$\kappa_1 = 1, 2, \ldots, q_1; \ \kappa_2 = 1, 2, \ldots, q_2; \ \ldots; \ \kappa_m = 1, 2, \ldots, q_m.$$

Correlation function is designed from the formula

$$K_{y_l}(t_1, t_2) = M\{[Y_l(t_1) - \overline{Y}_l][Y_l(t_2) - Y_l]\} \approx$$
$$\approx \sum_{\kappa_1\kappa_2\ldots\kappa_m} [y_l(t_1, V_{1\kappa_1}, V_{2\kappa_2}, \ldots, V_{m\kappa_m}) - \overline{Y}_l] \times$$
$$\times [y_l(t_2, V_{1\kappa_1}, V_{2\kappa_2}, \ldots, V_{m\kappa_m}) - \overline{Y}_l] \prod_{j=1}^{m} \rho_{\kappa_j}. \quad (V.15)$$

Page 129.

Dispersion is determined from the formula

$$D[Y_l(t)] \approx \sum_{\kappa_1\kappa_2\ldots\kappa_m} [y_l(t, V_{1\kappa_1}, V_{2\kappa_2}, \ldots, V_{m\kappa_m}) - \overline{Y}_l]^2 \times$$
$$\times \prod_{j=1}^{m} \rho_{\kappa_j}, \ \kappa_1 = 1, 2, \ldots, q_1; \ \kappa_2 \ \ 1, 2, \ldots, q_2; \ \ldots; \ \kappa_m \ \ 1, 2, \ldots, q_m.$$
$$(V.16)$$

The integral law of distribution [108] is designed from the formula

$$F(t, y_i) = P[Y_i(t) \leqslant y_i] \approx$$
$$\approx \sum_{\kappa_1 \kappa_2 \ldots \kappa_m} \frac{1}{2}\left[1 - \frac{y_i(t, V_{1\kappa_1}, V_{2\kappa_2}, \ldots, V_{m\kappa_m}) - y_i}{|y_i(t, V_{1\kappa_1}, V_{2\kappa_2}, \ldots, V_{m\kappa_m}) - y_i|}\right] \prod_{j=1}^{m} p_{\kappa_j}. \quad \text{(V.17)}$$

The total number of versions of the integrations of system of equations (V.1) is equal

$$N = q_1 q_2 \ldots q_m. \quad \text{(V.18)}$$

It follows from formula (V.18) that if total Mach number of random variables $V_j$ is great and values $q_1, q_2, \ldots, q_m$ are selected sufficiently large, then it is necessary to fulfill the large number N of numerical integrations of system (V.1).

However, if values $q_1, q_2, \ldots, q_m$ are chosen rationally, then it is possible to attain the required accuracy of the determination of the unknown probabilistic characteristics with the acceptable expenditures of machine time, which is necessary during the search for the optimum values of the parameters of the control system.

For the illustration let us give the formulas of interpolation method for calculating of mathematical expectations and dispersions of output coordinates for the specific cases during normal allocation of initial random variables.

Case m=1, q=2 (N=2). In this case into the right sides of the system enters altogether only one random variable with parameters a and $\sigma$, and two nodes (q=2) are chosen for calculating approximate values.

From Table 2 of Appendix 6 (with q=2) we obtain the values

$$\rho_{11}=0{,}5;\quad \rho_{12}=0{,}5;\quad \lambda_{11}=1;\quad \lambda_{12}=-1.$$

According to the formula of reduction (V.11) we have

$$V_{11}=a+\sigma;\ V_{12}=a-\sigma. \qquad \text{(V.19)}$$

Page 130.

It follows from formula (V.15) that in this case the mathematical expectations of the output coordinates

$$\mathrm{M}[y_i(t, V)] \approx y_i(t, a+\sigma)\,0{,}5+y_i(t, a-\sigma)\,0{,}5. \qquad \text{(V.20)}$$

Dispersion is designed from the formula

$$\begin{aligned} D[y_i(t)] \approx [y_i(t, a+\sigma)\,0{,}5-y_i(t, a-\sigma)\,0{,}5]^2\cdot 0{,}5+\\ +[y_i(t, a-\sigma)\,0{,}5-y_i(t, a+\sigma)\,0{,}5]^2\cdot 0{,}5=\\ =\frac{[y_i(t, a+\sigma)-y_i(t, a-\sigma)]^2}{4}. \end{aligned} \qquad \text{(V.21)}$$

It is not difficult to note that in this case formulas (V.2) and (V.21) completely coincide with expressions (IV.72) (IV.73), recorded for m=1, i.e., the method of equivalent perturbations and interpolation method give identical results.

Case m=1, q=8 (N=8). From table 2 of appendix 6 (with q=8) we obtain values $\rho_{11}$=0.3730122;

$\rho_{12}$=0.3730122; $\rho_{13}$=0.1172399; $\rho_{14}$=0.1172399;

$\rho_{15}$=0.009635220; $\rho_{16}$=0.009635220; $\rho_{17}$=0.0001126145;

$\rho_{18}$=0.00011264145; $V_{11}$=0.5390798; $V_{12}$=-0.5390798;

$V_{13}$=1.6365190; $V_{14}$=-1.6365190; $V_{15}$=2.8024859;

$V_{16}$=-2.8024859; $V_{17}$=4.1445472; $V_{18}$=-4.1445472.

Applying the formula of reduction (V.11), we will obtain

$$V_{1\kappa} = a_1 + \sigma_1\lambda_{1\kappa};\quad \kappa = 1, 2, \ldots, 8.$$

It follows from general formula (V.9) that the mathematical expectations of output coordinates in this case

$$M\left(y_i(t, V)\right] \approx \sum_{\kappa=1}^{8} \rho_{1\kappa} y_i(t, a_1 + \sigma_1\lambda_{1\kappa}), \qquad \text{(V.22)}$$

where numbers $\rho_{1\kappa}$ and $\lambda_{1\kappa}$ have values indicated above, and

$$a_1 = M[V]; \quad \sigma_1 = \sqrt{D[V]}.$$

Case m=3; $q_1$=2; $q_2$=$q_3$=1; N(=$q_1q_2q_3$=2).

From Table 2 of Appendix 6, just as with m=1, we obtain values $\rho_{11}$=0.5; $\rho_{12}$=0.5; $\lambda_{11}$=1; $\lambda_{12}$=-1. For the random variable $V_2$ we have $\rho_{21}$=1; $\lambda_{21}$=0.000. For the random variable $V_3$ we have $\rho_{31}$=1; $\lambda_{31}$=0.000.

Page 131.

Applying the formulas of reduction, we will obtain

$$\left.\begin{aligned} &\lambda_{1\kappa_1} = \bar{\lambda}_1 + \sigma_1 V_{1\kappa_1}; && \kappa_1 = 1, 2; \\ &\lambda_{2\kappa_2} = \bar{\lambda}_2; && \kappa_2 = 1; \\ &\lambda_{3\kappa_3} = \bar{\lambda}_3; && \kappa_3 = 1. \end{aligned}\right\} \qquad \text{(V.23)}$$

Substituting the values of numbers $\rho_{j\varkappa_j}$ and $V_{1\varkappa_1}, V_{2\varkappa_2}, V_{3\varkappa_3}$ indicated into formulas (V.14) (V.16), we will obtain for calculating the mathematical expectations of output coordinates the following formula:

$$M[y_i(t, V_1, V_2, V_3)] \approx \sum_{\varkappa_1=1}^{2} \rho_{1\varkappa_1}\rho_{21} y_i(t, V_{1\varkappa_1}, V_{21}, V_{31}) = \\ = \frac{1}{2}[y_i(t, a_1+\sigma_1, a_2, a_3) + y_i(t, a_1-\sigma_1, a_2, a_3)], \quad (V.24)$$

where $a_1$, $a_2$, $a_3$ - values of the mathematical expectations of the random variables $V_1$, $V_2$, $V_3$;

$\sigma_1$ - the root-mean-square deviation of the random variable $V_1$;

$$D[y_i(t, V_1, V_2, V_3)] = \frac{[y_i(t, a_1+\sigma_1, a_2, a_3) - y_i(t, a_1-\sigma_1, a_2, a_3)]^2}{4}.$$

Case m=3; $q_1=q_2=q_3=2$ ($N=q_1q_2q_3=8$). they coincide:

The mathematical expectations of output coordinates in this case

are calculated according to the formula

$$\begin{aligned}M[y_i(t, V_1, V_2, V_3)] \approx (0,5)^3 [y_i(t, a_1+\sigma_1, a_2+\sigma_2, a_3+\sigma_3) + \\ + y_i(t, a_1-\sigma_1, a_2+\sigma_2, a_3+\sigma_3) + y_i(t, a_1+\sigma_1, a_2-\sigma_2, a_3+\sigma_3) + \\ + y_i(t, a_1+\sigma_1, a_2+\sigma_2, a_3-\sigma_3) + y_i(t, a_1-\sigma_1, a_2-\sigma_2, a_3+\sigma_3) + \\ + y_i(t, a_1+\sigma_1, a_2-\sigma_2, a_3-\sigma_3) + y_i(t, a_1-\sigma_1, a_2+\sigma_2, a_3-\sigma_3) + \\ + y_i(t, a_1-\sigma_1, a_2-\sigma_2, a_3-\sigma_3)].\end{aligned}$$

Case $m=3$, $q_1=3$, $q_2=2$, $q_3=1$ ($N=q_1q_2q_3=6$).

From table 2 of appendix 6 for $q_1=3$ we have

$\rho_{11}=0.6666667$; $\rho_{12}=0.1666667$; $\rho_{13}=0.1666667$; $\lambda_1=0.000000$;

$\lambda_{12}=1.7320508$; $\lambda_{13}=-1.7320508$;

for $q_2=2$, we have

$$\rho_{21}=0,5;\quad \rho_{22}=0,5;\quad \lambda_{21}=1;\quad \lambda_{22}=-1;$$

for $q_3=1$ we have

$$\rho_{31}=1;\quad \lambda_{31}\quad 0.$$

Page 132.

Using formulas of reduction, we will obtain

$$\left.\begin{array}{ll} V_{1\kappa_1} = a_1 + \sigma_1\lambda_{1\kappa_1}; & \kappa_1 = 1, 2, 3; \\ V_{2\kappa_2} = a_2 + \sigma_2\lambda_{2\kappa_2}; & \kappa_2 = 1, 2; \\ V_{3\kappa_3} = a_3, & \kappa_3 = 1. \end{array}\right| \qquad (V.25)$$

From the general formulas for calculating of mathematical expectations and dispersions we obtain

$$\begin{aligned} M[y_i(t, V_1, V_2, V_3)] \approx \sum_{\kappa_1\kappa_2\kappa_3} y_i(t, V_{1\kappa_1}, V_{2\kappa_2}, V_{3\kappa_3}) \prod_{j=1}^{3} \rho_{j\kappa_j} = \\ = \sum_{\kappa_1\kappa_2} y_i(t, V_{1\kappa_1}, V_{2\kappa_2}, a_3) \prod_{j=1}^{2} \rho_{j\kappa_j} = y_i(t, V_{11}, V_{21})\rho_{11}\rho_{21} + \\ + y_i(t, V_{11}, V_{22})\rho_{11}\rho_{22} + y_i(t, V_{12}, V_{21})\rho_{12}\rho_{21} + \\ + y_i(t, V_{12}, V_{22})\rho_{12}\rho_{22} + y_i(t, V_{13}, V_{21})\rho_{13}\rho_{21} + \\ + y_i(t, V_{13}, V_{22})\rho_{13}\rho_{22}, \end{aligned} \qquad (V.26)$$

where numbers $V_{1\kappa_1}, V_{2\kappa_2}, V_{3\kappa_3}$ are determined with the help of the tabular values according to formulas (V.11), and numbers $\rho_{1\kappa_1}, \rho_{2\kappa_2}, \rho_{3\kappa_3}$ are equal to indicated above.

4. Reduction of the distribution laws to the tabular cases.

The formulas of interpolation method, which allow with the help of the tables of Chebyshev's nodes and Christoffel number to solve the problem of determining the statistical characteristics of automatic control systems for the most frequently encountered laws of

distribution of the probabilities of initial random variables, were given in p. 3. Has sense to consider also the possibility of applying the interpolation method under the laws of distribution of the probabilities, different from the uniform, the normal or the exponential.

It is necessary to note that sometimes the task of the construction of the tables of the nodes of Chebyshev and Christoffel numbers can be solved analytically.

Page 133.

There is practical interest in the law of distribution of Chebyshev

$$p(v) = \frac{1}{\pi\sqrt{1-v^2}}, \; v \in [-1, 1],$$

which is called also the law of "arc sine", since its integral form takes the form

$$P(v) = \int_{-1}^{v} \frac{dz}{\pi\sqrt{1-z^2}} = \frac{1}{2} + \frac{1}{\pi}\arcsin v. \qquad \text{(V.27)}$$

For this law it is possible to indicate the analytical formulas, which express the nodes of Chebyshev and Christoffel numbers explicitly:

$$\left.\begin{array}{c} V_\kappa = \cos\frac{(2\kappa - 1)\pi}{2q}, \ \kappa = 1, 2, \ldots, q; \\ \rho_\kappa \quad \frac{1}{q}, \end{array}\right\} \quad (V.28)$$

where q - total number of nodes from parameter $v_\kappa$.

In order to use the existing tables of nodes or formula (V.28), is expediently to preliminarily determine certain nonlinear conversion of the form

$$\lambda \quad \psi(v), \qquad (V.29)$$

the bringing specified distribution of probabilities to any of allocations indicated above.

It is known that function $\psi(v)$ it is determined by the equation

$$P_v(v) \quad P_\lambda[\psi(v)], \qquad (V.30)$$

where $P_\lambda$ — integral law of distribution of the probabilities of random variable $\lambda$;

$P_v(v)$ — integral law of random number distribution, for which the tables of Chebyshev's nodes are (or to eat analytical formulas).

Let us give the example, which illustrates the use/application of formula (V.30).

Let

$$P_\lambda = 1 - e^{-\frac{\lambda^2}{2D}};\ D > 0,\ \lambda \in [0, \infty), \qquad (V.31)$$

i.e. the random variable λ have a law of Rayleigh distribution. As standard allocation let us look at the exponential law of distribution, for which

$$P_v(v) = 1 - e^{-v},\ v \in [0, \infty). \qquad (V.32)$$

In table 2 of appendix 6 for the law of distribution (V.32) are values of nodes of Chebyshev and numbers of Christoffel.

Page 134.

From equation (V.30) it follows that

$$1 - e^{-v} = 1 - e^{-\frac{\lambda^2}{2D}};$$

hence

$$v = \frac{\lambda^2}{2D}.$$

Consequently, in this example

$$\lambda = \psi(v) = \sqrt{2Dv}. \qquad \text{(V.33)}$$

Obtained conversion (V.33) makes it possible to carry out transition/junction from Rayleigh's law to the standard form of exponential law. Formula (V.33) can be used also if necessary to continue or to make more precise the available tables of nodes of the type of Chebyshev and numbers of Christoffel.

As an example let us consider the standard case of the uniform distribution of the probabilities, when

$$P_\lambda = \frac{\lambda+1}{2}, \ \lambda \in [-1, 1]. \qquad \text{(V.34)}$$

Let us find φ-transform which reduces uniform law to Chebyshev's law

$$P_v = \frac{1}{2} + \frac{1}{\pi} \arcsin v. \qquad \text{(V.35)}$$

It follows from equations (V.34) and (V.35) that

$$\frac{1}{2} + \frac{1}{\pi} \arcsin v = \frac{\lambda+1}{2}.$$

Therefore the conversion of the form

$$\lambda = \frac{2}{\pi} \arcsin v \qquad (V.36)$$

will lead the uniform law (in the standard case) to Chebyshev's law. Applying this $\psi$-conversion, during calculations it is possible to use analytical formulas (V.28) for the determination of the nodes of Chebyshev and numbers of Christoffel for Chebyshev's laws.

If random variable λ has the nonstandard form of the uniform law of distribution (λ ∈ [a, b]), then in this case the required $\psi$-conversion is expressed by the formula

$$\lambda = \frac{b+a}{2} + \frac{b-a}{\pi} \arcsin v. \qquad (V.37)$$

Conversion the bringing arbitrary exponential of allocation $P_\lambda = 1 - e^{-a\lambda}$, $a > 0$, $\lambda \in [0, \infty)$ to allocation of Chebyshev, is determined it is analogously and expressed by the following formula:

$$\lambda = -a \ln\left(\frac{1}{2} - \frac{1}{\pi} \arcsin v\right). \qquad (V.38)$$

Page 135.

Conversion giving normal allocation to allocation of Chebyshev, does not succeed in obtaining explicitly, since the integral distribution law is expressed as the Laplace function.

It should be noted that $\psi$-conversion inputs/embeds further nonlinearity, which leads to an increase in the number of nodes of interpolation for achievement of the preset accuracy.

5. Selection of optimum step/pitch for the numerical integration of the differential equations of those containing the random parameters.

During the computation of the probabilistic characteristics of automatic control system with the help of the computer(s) the need for the rational selection of value h of the step/pitch of numerical integration appears.

If the overall gap/interval of integration has a length T, then for obtaining each particular solution it is necessary to do T/h steps/pitches of numerical integration.

Let the method of numerical integration, which requires for the fulfillment of one step/pitch A of standard computer operations, be

selected. In this case the overall volume of computational work for obtaining N the realizations

$$M = AN\frac{T}{h}. \qquad (V.39)$$

It follows from this formula that with an increase in the step/pitch of numerical integration the overall volume of computational work proportionally is reduced. However, in this case increases also an error in the result, which is not always acceptable.

Thus, appears the task of the selection of the optimum step/pitch of the numerical integration, which ensures the preset accuracy of calculations with the smallest volume of computational work.

The accuracy of result is usually rated/estimated by the value of the relative error

$$R(h) = \sup_i \frac{\sup_t |\tilde{y}_i(t) - y_i(t)|}{\sup_t |y_i(t)|}, \qquad (V.40)$$

where $\tilde{y}_i(t)$ — approximate value of output coordinate $y_i(t)$, obtained by the method of numerical integration; $y_i(t)$ — exact value of output coordinate $y_i$.

Symbol $\sup_t[\quad]$ indicates the operation of the selection of the greatest value of relative error for all t from interval $t\in[0, T]$.

Page 136.

Requirement on the accuracy of numerical integration can be assigned in the form of the greatest permissible value of relative error $\varepsilon_{gi}$. In this case the accuracy of results will be not below required, if the inequalities

$$R_i(h)\leqslant\varepsilon_{gi} \qquad (V.41)$$

are fulfilled simultaneously for all i=1, 2, ..., n. During the selection of value $\varepsilon_{gi}$ it is possible to be guided by following rule of thumb.

If it is necessary as the final result to determine probabilistic characteristics with an accuracy to two significant places, then one should take $\varepsilon_{gi}=0,001$, if to three significant places, then $\varepsilon_{gi}=0,0001$ and so forth.

Thus, values $\varepsilon_{gi}$ must have after the comma as many zero, as is required to obtain the significant places for the probabilistic characteristic in question. It must be noted that usually value $\varepsilon_{gi}$ is chosen for all i of one and the same.

The task of the selection of the optimum step/pitch of integration mathematically can be formulated as follows.

To find value $h_0$, with which will be satisfied the conditions

$$\left.\begin{aligned} R_i(h_0) &\leqslant \varepsilon_{gi}; \\ M(h_0) = AN\frac{T}{h} &= \min_{h>0} M(h). \end{aligned}\right\} \qquad (V.42)$$

The exact solution of this task requires greater computations than strictly numerical integration; therefore is proposed the approximate method of the solution of this problem.

In the system of differential equations (V.1), which describe the dynamics of automatic control system, the differential equations, which correspond to the most high speed (least inertial) components/links, are selected. The characteristics of the nonlinear elements/cells of these components/links (or one component/link) are replaced by their linear approximations, and input variables - by constant values.

Integrating obtained thus linear differential equation with the constant coefficients, we will obtain analytical formulas for the solutions and their derivatives. Since these formulas determine approximately most oscillating component, then it is possible to use them for the estimate of the magnitude R(h) of an absolute error in the numerical integration depending on the value of step/pitch. Value $h_*$ we choose greatest among those h, for which is fulfilled the inequality

$$R_i(h) \leqslant \varepsilon_{gi}. \qquad \text{(V.43)}$$

It is possible to obtain calculation formulas for different methods of numerical integration by method presented above.

Page 137.

Let us designate the order of an error in the method of numerical integration by symbol p.

If we disregard/neglect the effect of round-off errors and the possible accumulation of error upon transfer from one step/pitch to the next, then an absolute error in the numerical integration can be

rated/estimated according to the formula

$$y(t+h)-\tilde{y}(t+h)\approx\frac{y^{(p+1)}(t+\alpha(t)h)}{(p+1)!}h^{p+1}. \qquad (V.44)$$
$$0\leqslant\alpha(t)\leqslant 1$$

In this case for t in question relative error is determined by the formula

$$R(t,h)\approx\frac{h^{p+1}}{(p+1)!}\left|\frac{y^{(p+1)}(t+\alpha(t)h)}{\max\limits_t|y(t)|}\right|, \qquad (V.45)$$

where $y^{(p+1)}$ — derivative of the (p+1)-th order.

Therefore for interval of t it is possible to obtain the evaluation/estimate

$$R(h)\approx\frac{h^{p+1}}{(p+1)!}\max_t\left|\frac{y^{(p+1)}(t+\alpha(t)h)}{\max\limits_t|y(t)|}\right|. \qquad (V.46)$$

From inequality (V.43) we obtain the approximation formula for the optimum value of the step/pitch

$$h_0\approx\sqrt{\varepsilon_g(p+1)!\,\frac{\max\limits_t|y(t)|}{\max\limits_t|y^{(p+1)}(t)|}}. \qquad (V.47)$$

Most frequently least inertial components/links are represented

in the form of the aperiodic component/link of the first or second order or oscillating circuit/member of the second order.

These components/links have components of the form

$$y(t) = Ce^{\lambda t}, \qquad \text{(V.48)}$$

where C - certain constant, in the general case a complex number;

λ - root of the characteristic equation of component/link.

It follows from formula (V.48) that

$$\max_t |y^{(p+1)}(t)| = \lambda^{p+1} \max_t |y(t)|. \qquad \text{(V.49)}$$

Substituting expression (V.49) in relationship/ratio (V.47), we will obtain the simple calculation formula

$$h_0 \approx \frac{1}{|\lambda|^{p+1}} \sqrt{\varepsilon_g (p+1)!}. \qquad \text{(V.50)}$$

Page 138.

Value |λ|, entering this formula, is the absolute value of the root with the maximum absolute value among the roots of the characteristic equations of the chosen components/links of system.

Example. Let be isolated two components/links, described by the differential equations

$$\frac{dy_1}{dt}+3y_1=A_1;$$
$$\frac{d^2y_2}{dt^2}+4\frac{dy_2}{dt}+6y_2=A_2.$$

where $A_1$ and $A_2$ - certain values.

In order to compute value $h_e$ according to formula (V.50), it is necessary to solve the appropriate characteristic equations

$$\lambda+3=0;$$
$$\lambda^2+4\lambda+6=0.$$

Roots of these equations

$$\lambda_1=-3;\quad \lambda_2=2+i\sqrt{2};\quad \lambda_3=2-i\sqrt{2}.$$

We find their absolute values:

$$|\lambda_1|=3;\quad |\lambda_2|=|\lambda_3|=\sqrt{6}.$$

Since $\sqrt{6}<3$, then $|\lambda|=3$. For Euler's method p=1; therefore when

$\varepsilon_t$ =0.0001 we will have

$$h_0 \approx \frac{1}{3}\sqrt{0,0001\cdot 2} = \frac{\sqrt{2}}{3}0,01 \approx 0,005.$$

For the method of Runge-Kutta of fourth order (p=4) we will obtain

$$h_0 = \frac{1}{3}\sqrt[5]{0,0001\cdot 5!} = \frac{1}{3}\sqrt[5]{0,012} \approx 0,14.$$

Analogously it is possible to calculate value $h_0$, also, for other methods of numerical integration.

If the right sides of the reference system of differential equations have first-order discontinuities, then the methods, which give high accuracy only with the sufficiently flat right sides (methods of Runge-Kutta, Adams, etc.), to apply inexpediently. In these cases it is necessary to use the method of Euler or his simplest modifications. With discontinuous right sides independent of the method of numerical integration used the step/pitch of numerical integration must be designed with p=1.

Page 139.

The essential decrease of the volume of computational work it is

possible to attain due to the separation of the system of differential equations into the series/row of subsystems with different inertness, so that for the more inertial systems in accordance with formula (V.50) it would be possible to select the larger step/pitch of numerical integration.

However, the need for development for each class of the systems of the specific routines of numerical integration somewhat decreases the efficiency of this method.

The space of computations can be shortened also due to the separation of the gap/interval of integration into several parts so that on some of them it would be possible to conduct integration with the steep pitch of integration, than the accepted for formula (V.50) collective pitch.

6. Task of the selection of the necessary number of nodes.

As it was established/installed in p. 3 of this chapter, total number $N = q_1 q_2 \ldots q_m$ of all versions of integration, necessary for computing the probabilistic characteristics, depends on the selection of values $q_1, q_2, \ldots, q_m$. Value N enters by multiplier into formula (V.39), which determines the overall volume of the necessary computational work.

Consequently, for decreasing the volume of the computational work it is necessary of value $q_1, q_2, \ldots, q_m$ to choose so, in order to number N minimum when the required accuracy of the computations of probabilistic characteristics is provided.

The exact solution of the task of the optimum selection of the necessary number of nodes of integration in the general case requires the considerably larger space of computations, than calculations regarding the unknown probabilistic characteristics. Therefore practical value have the approximation methods of the selection of a number of nodes of integration, with which the space of computations proves to be comparatively small.

It is necessary to keep in mind that with an increase in the number of nodes a quantity of versions $N = q_1 q_2 \ldots q_m$ of integrations increases according to the multiplicative law, and together with N increases also the volume of computational work.

If one assumes that values $q_1, q_2, \ldots, q_m$ affect the accuracy of result independently of each other, then the smallest value of value N will be obtained when each of the values $q_k$ is selected minimum when the required accuracy of result is provided. Under this assumption the

task of determining the optimum values of values $q_\kappa$ breaks down into m independent tasks regarding smallest possible degree $m_\kappa$ of certain polynomial relative to random variable $V_\kappa$ (other random variables in this case they are assumed to be those fixed/recorded), sufficient which approaches the preset characteristic function. It is first of all necessary to isolate random variables, for which $q_\kappa = 1$.

Page 140.

In this case let us determine those random variables, on which the function $\chi$ depends linearly:

$$\chi = A + \sum_{\kappa=1}^{m} A_\kappa V_\kappa, \qquad \text{(V.51)}$$

where $A, A_\kappa$ — some constant numbers.

Then

$$M[\chi] = A + \sum_{\kappa=1}^{m} A_\kappa M[V_\kappa]. \qquad \text{(V.52)}$$

In the general case

$$\chi = A + \sum_{\kappa=1}^{m} A_\kappa V_\kappa + \sum_{\kappa_1=1}^{m} \sum_{\kappa_2=1}^{m} A_{\kappa_1\kappa_2} V_{\kappa_1} V_{\kappa_2} + \sum_{\kappa_1=1}^{m} \sum_{\kappa_2=1}^{m} \sum_{\kappa_3=1}^{m} \times$$
$$\times A_{\kappa_1\kappa_2\kappa_3} V_{\kappa_1} V_{\kappa_2} V_{\kappa_3} + \ldots + \sum_{\kappa_1=1}^{m} \ldots \sum_{\kappa_q=1}^{m} \times$$
$$\times A_{\kappa_1 \ldots \kappa_q} V_{\kappa_1} \ldots V_{\kappa_q} + \varepsilon, \qquad \text{(V.53)}$$

where $q = \max m_\kappa$;

$m_\kappa$ — the maximum value of degree relative to random parameter $V_\kappa$.

It is possible to assume that from first s of random variables $V_1, V_2, \ldots, V_s$ the function $\chi$ in the limits of the preset accuracy depends only linearly. Then this function can be represented in the form

$$\chi = A + \sum_{\kappa=1}^{m} A_\kappa V_\kappa + \psi(V_{s+1}, V_{s+2}, \ldots, V_m) + \varepsilon, \quad (V.54)$$

where $\psi$ - certain nonlinear function relative to random variables $V_{s+1}, V_{s+2}, \ldots, V_m$;

hence

$$M[\chi] = A + \sum_{\kappa=1}^{m} A_\kappa a_\kappa + M[\psi + \varepsilon]; \quad (V.55)$$

from this formula it follows that with an increase in the number of nodes from parameters $V_1, V_2, \ldots, V_s$ result (V.55) must remain virtually constant/invariable.

Analogously discussing, it is possible to consecutively/serially determine random variables, for which one should take q=2, then it consists of the following stages:

1. Calculation of value M[$\chi$] with N=1 and $q_1 = q_2 = \ldots q_m = 1$.

In this case random variables $V_1, V_2, \ldots, V_m$, entering the right sides of the system of differential equations (V.1), are replaced by values of their mathematical expectations $\bar{V}_1, \bar{V}_2, \bar{V}_m$. Value $M_0[\chi]$ is obtained as a result.

Page 141.

2. Value M[$\chi$] is designed with N=2 for following m of versions:

$$1)\ q_1 = 2,\ q_2 = q_3 = \ldots = q_m = 1$$
$$(V_{11} = u_1 + \sigma_1,\ V_{12} = u_1 - \sigma,\ V_2 = u_2,\ V_3 = u_3, \ldots, V_m = u_m);$$
$$2)\ q_1 = 1;\ q_2 = 2;\ q_3 = q_4 = \ldots = q_m = 1$$
$$(V_1 = u_1,\ V_{21} = u_2 + \sigma,\ V_{22} = u_2 - \sigma_2,\ V_3 = u_3, \ldots, V_m = u_m);$$
$$m)\ q_1 = q_2 = \ldots = q_{m-1} = 1;\ q_m = 2$$
$$(V_1 = u_1,\ V_2 = a_2, \ldots, V_{m-1} = u_{m-1},\ V_{m1} = u_m + \sigma,\ V_{m2} = u_m - \sigma).$$

Values

$$M_1[\chi],\ M_2[\chi], \ldots, M_m[\chi].$$

are obtained as a result.

3. Comparison of values $M_1[\chi], M_2[\chi], \ldots, M_m[\chi]$, obtained in p. 2 with value of $M_0[\chi]$, obtained in p. 1.

If

$$|M_\kappa[\chi] - M_0[\chi]| \leqslant \varepsilon,$$

where $\epsilon$ - value of the maximum permissible error, then $q_\kappa = 1$; if

$$|M_\kappa[\chi] - M_0[\chi]| > \varepsilon,$$

value $q_\kappa$ it is necessary to choose more than one.

4. Calculation of value $M_0[\chi]$ with $N = 2^{m-s}$.

$$\begin{aligned}
&1)\ q_{s+1} = 1;\ q_{s+2} = 1; \ldots ;\ q_m = 1;\\
&\quad q_1 = 1;\ q_2 = 1; \ldots ;\ q_s = 1;\\
&2)\ q_{s+1} = 2;\ q_{s+2} = 1; \ldots ,\ q_m = 1;\\
&\quad q_1 = 1;\ q_2 = 1; \ldots ;\ q_s = 1;\\
&3)\ q_{s+1} = 1;\ q_{s+2} = 2; \ldots ;\ q_m = 1;\\
&\quad q_1 = 1;\ q_2 = 1; \ldots ;\ q_s = 1;\\
&4)\ q_{s+1} = 2;\ q_{s+2} = 2;\ q_m = 1;\\
&\quad q_1 = 1;\ q_2 = 1; \ldots ;\ q_s = 1;\\
&\quad \ldots\ldots\ldots\ldots\ldots\\
&2^{m-s})\ q_{s+1} = 2;\ q_{s+2} = 2; \ldots ;\ q_m = 2;\\
&\quad q_1 = 1;\ q_2 = 1; \ldots ;\ q_s = 1.
\end{aligned}$$

Page 142.

5. Calculation of values $M_\varkappa[\chi]$ for "m-s" versions:

$$\begin{aligned}
&1)\ q_{s+1}=3;\ q_{s+2}=2, \dots ;\ q_m=2;\\
&\quad q_1=1;\ q_2=1, \dots ;\ q_s=1;\\
&2)\ q_{s+1}=2;\ q_{s+2}=3;\ q_{s+3}=2, \dots ;\ q_m=2;\\
&\quad q_1=1;\ q_2=1; \dots ;\ q_s=1;\\
&\text{«}m-s\text{»})\ q_{s+1}=q_{s+2}=\dots=q_{m-1}=2;\ q_m=3;\\
&\quad q_1=1;\ q_2=1, \dots ;\ q_s=1.
\end{aligned}$$

6. Comparison of values $M_\varkappa[\chi]$, obtained in p. 5, with value of $M_0[\chi]$, obtained in point/item 4.

If

$$|M_\varkappa[\chi]-M_0[\chi]|\leqslant\varepsilon,$$

then for such indices to $q_\varkappa=2$; otherwise $q_\varkappa>2$.

7. Values of $M_e[\chi]$ for $N = 2^p 3^{m-s-p}$: are again recalculated:

$$q_1 = q_2 = \ldots = q_s = 1;\ q_{s+1} = q_{s+2} = \ldots = q_{s+p} = 2;$$
$$q_{s+p+1} = \ldots = q_m = 3;$$

calculation of m-(s+p) versions with a consecutive increase in the number of nodes by one, comparison of results with selection $q_\kappa = 3$ and so forth.

Calculation is finished, when values $q_\kappa$ for all indices $\kappa$=1, 2, ..., m are determined.

The algorithm of the selection of a number of nodes of interpolation indicated it is expedient to apply when the repeated computation of the preset probabilistic characteristic is required, which, for example, occurs at the solution of the problem of the search for the optimum parameters of automatic control system.

It is possible to determine approximately in single cases values $q_1, q_2, \ldots, q_m$ analyzing the structure of the right sides of the preset differential equations.

In this case just as in the task of the selection of the optimum step/pitch of numerical integration, most effective prove to be the

approximate analytical methods. In this case from the physical considerations it is possible to isolate the components/links of the first or second orders, which to the greatest degree are subjected to the effect of the random factors in question in the automatic control system.

Replacing the nonlinear characteristics of components/links by their linear approximations, we will obtain for the components/links linear differential equations with the constant coefficients, whose solutions can be recorded in the form of known formulas.

Page 143.

Analyzing these formulas, it is possible to determine the necessary degrees of the approximating polynomials.

During the optimum selection of the nodes of interpolation the accuracy of result with an increase in the number of nodes grows (or it remains constant/invariable). Therefore the reached accuracy can be checked by the method of comparison of the results, obtained at selected values $q_*$ and with a larger number of nodes.

During the selection of values $q_*$ it is necessary to keep in mind that for computing the dispersions of output coordinates is

required a doubly larger number of nodes, than during the computation of the mathematical expectations of output coordinates.

7. Use/application of a method of non-canonical expansions to the task of the construction of the realizations of the random functions, which simulate external disturbances/perturbations.

Stationary case. For the practical use of an interpolation method more the value has a reduction of random functions to the system of random variables. Let us consider one of the methods of solution of this task.

Let us assume that random function the X(t) stationary and has mathematical expectation, equal to $m_X(t)$, and the correlation function

$$R_X(t_1, t_2) = R_X(\tau),$$

where $\tau = t_2 - t_1$.

The laws of distribution of probabilities for X(t) can be unknowns. It is necessary to indicate the method of the construction of the realizations of random function X(t) according to its preset probabilistic characteristics $m_X(t)$ and $R_X(\tau)$.

As the criterion of the conformity of the constructed realizations to the preset probabilistic characteristics let us take the requirement of the identity of mathematical expectations and correlation functions. In this case it is possible to speak about the identity of random functions in the limits of correlation theory.

Let us consider the random function

$$\tilde{X}(t) = m_X(t) + V_1 \cos \omega t + V_2 \sin \omega t, \qquad (V.56)$$

where $V_1$, $V_2$, $\omega$ - some random variables.

It is possible to demonstrate that during the specific selection of the probabilistic characteristics of the random variables $V_1$, $V_2$ and $\omega$ random function $\tilde{X}(t)$ will have mathematical expectation, equal to preset function $m_X(t)$, and the correlation function, equal to preset function $R_X(\tau)$.

Page 144.

In this case it will be established/installed, that random functions $\tilde{X}(t)$ and X(t) identically coincide in the limits of correlation theory, and therefore during the construction of realizations for random function X(t) it is possible to use formula (V.56).

The construction of realizations according to formula (V.56) does not cause special difficulties. Necessary for this operation values of the random variables $V_1$, $V_2$ and $\omega$ can be easily obtained with the help of different algorithms for the generation of the random or pseudo-random numbers (see Chapter XI). During the use of an interpolation method the necessary values of random variables are designed from formulas (V.11) with the help of the special tables.

Let us pass to the proof.

Let us assume that the random variables $V_1$, $V_2$ and $\omega$ are independent, moreover the mathematical expectations of the random variables $V_1$ and $V_2$ are equal to zero.

Then

$$M[\tilde{X}(t)] = M[m_x(t) + V_1 \cos \omega t + V_2 \sin \omega t] =$$
$$= m_x(t) + M[V_1]\,M[\cos \omega t] + M[V_2]\,M[\sin \omega t] = m_x(t),$$

since by hypothesis $M[V_1]=M[V_2]=0$.

Thus established/installed, that

$$M[\tilde{X}(t)] = m_x(t) = M[X(t)]. \qquad (V.57)$$

Let us assume

$$R_X(\tau) = \sigma_x^2 r(\tau), \qquad \text{(V.58)}$$

where $\sigma_x^2 = D[X]$ — the dispersion of random function X(t);

r($\tau$) - the normalized correlation function.

Let us compute $R_{\tilde{X}}(\tau)$:

$$\begin{aligned} R_{\tilde{X}}(\tau) &= \mathrm{M}\,[(\tilde{X}(t_1) - m_X(t_1))(\tilde{X}(t_2) - m_X(t_2))] \\ &= \mathrm{M}\,[(V_1 \cos\omega t_1 + V_2 \sin\omega t_1)(V_1 \cos\omega t_2 + V_2 \sin\omega t_2)] = \\ &= \mathrm{M}\,[V_1^2]\,\mathrm{M}\,[\cos\omega t_1 \cos\omega t_2] + \mathrm{M}\,[V_2^2]\,\mathrm{M}\,[\sin\omega t_1 \sin\omega t_2]. \end{aligned}$$

If one assumes that

$$\mathrm{M}\,[V_1^2] = \mathrm{M}\,[V_2^2] = \sigma_x^2, \qquad \text{(V.59)}$$

then

$$\begin{aligned} R_{\tilde{X}}(\tau) &= \sigma_x^2 \mathrm{M}\,[\cos\omega t_1 \cos\omega t_2 + \sin\omega t_1 \times \\ &\times \sin\omega t_2] = \sigma_x^2 \mathrm{M}\,[\cos\omega\tau]. \end{aligned} \qquad \text{(V.60)}$$

From the condition of the identity of the correlation functions

$$R_{\tilde{X}}(\tau) \equiv R_X(\tau) \qquad \text{(V.61)}$$

we obtain the equation

$$R_x(\tau) = \sigma_x^2 M[\cos\omega\tau]. \quad (V.62)$$

Page 145.

From formulas (V.58) (V.62) and the formula

$$M[\cos\omega\tau] = \int_{-\infty}^{\infty} (\cos\omega\tau) p(\omega)\, d\omega \quad (V.63)$$

we obtain the equation

$$r(\tau) = \int_{-\infty}^{\infty} (\cos\omega\tau)\, p(\omega)\, d\omega \quad (V.64)$$

for determining the unknown function $p(\omega)$, which is according to the sense the density of the distribution of the probabilities of random variable $\omega$.

Equation (V.64) is integral. However, it easily is solved, if one assumes that density $p(\omega)$ is sought in the class of even functions.

In this case

$$\int_{-\infty}^{\infty} (\sin \omega\tau)\, p(\omega)\, d\omega = 0. \qquad (V.65)$$

Therefore integral equation (V.64) can be represented in the form

$$r(\tau) = \int_{-\infty}^{\infty} (\cos \omega\tau)\, p(\omega)\, d\omega + j \int_{-\infty}^{\infty} (\sin \omega\tau) p(\omega) \times d\omega = \int_{-\infty}^{\infty} e^{j\omega\tau} p(\omega)\, d\omega. \qquad (V.66)$$

It follows from equation (V.66) that the normalized correlation function $r(\tau)$ is Fourier transform density $p(\omega)$.

From the theory of random functions it is known that the correlation function of any stationary random process is connected with its spectral density with Fourier transform in the form

$$\left.\begin{aligned} S_X(\omega) &= \int_{-\infty}^{\infty} e^{-j\omega\tau} R_X(\tau)\, d\tau; \\ R_X(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega\tau} S_X(\omega)\, d\omega, \end{aligned}\right\} \qquad (V.67)$$

where $S_x(\omega)$ — the spectral density of random function X(t).

It follows from formulas (V.58) and (V.67) that

$$r(\tau)=\int_{-\infty}^{\infty} e^{j\omega\tau}\left[\frac{S_X(\omega)}{2\pi\sigma_x^2}\right]d\omega. \qquad \text{(V.68)}$$

Page 146.

Since Fourier transform is unambiguous, it follows from equation (V.66) and formula (V.68) that the unknown density of the distribution of probabilities is equal to

$$p(\omega)=\frac{S_X(\omega)}{2\pi\sigma_x^2}. \qquad \text{(V.69)}$$

Let us note that obtained solution (V.69) of integral equation (V.66) has all properties of the density of the distribution of probabilities.

Is actual/real, $p(\omega)\geq 0$, since $S_x(\omega)\geqslant 0$;

$$\int_{-\infty}^{\infty} p(\omega)\,d\omega=1,$$

since

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\,d\omega \quad \sigma_x^2.$$

Thus, if the random variables $V_1$, $V_2$ and $\omega$ are independent and their mathematical expectations are equal to zero, and dispersions are equal to the random function X(t) preset to dispersion, then it is possible to indicate the density of random number distribution $\omega$ according to formula (V.69), such, which $\tilde{X}(t)$ will present the stationary random function X(t) taking into account the identity of the first two moments/torques.

This fact is a sufficient proof of method indicated above of the construction of the realizations of random function X(t).

In contrast to the canonical expansion, which represents random function X(t) in the form of series/row with an infinite number of random variables, the representation of random function X(t) according to formula (V.56), which we will call non-canonical, contains altogether only three auxiliary random variables.

Let us note that the number of random variables, entering the non-canonical expansion X(t), can be reduced up to two. For this it

is necessary to use the formula

$$\tilde{X}(t) = m_x(t) + \sigma_x(\cos\omega t + V\sin\omega t), \qquad (V.70)$$

where it is proposed that the random variables V and $\omega$ are independent, the mathematical expectation of value V is equal to zero, dispersion is equal to one, and the probability density of random variable $\omega$ is determined from formula (V.69). Proof is conducted analogously.

For facilitating the practical use/application let us give some results of the analytical calculations of density $p(\omega)$ for the correlation functions, utilized in the theory of automatic control.

Page 147.

Example 1. If

$$r_x(\tau) = e^{-\frac{\sigma^2\tau^2}{2}},$$

then

$$p(\omega) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{\omega^2}{2\sigma^2}}, \qquad (V.71)$$

i.e. in this example random variable $\omega$ must have the normal

distribution law.

Example 2. If

$$r_X(\tau)=\frac{\sin a\tau}{a\tau},\ a>0,$$

then

$$p(\omega)=\begin{cases}\frac{1}{2a},\omega\in[-a,a],\\ 0,\omega\notin[-a,a],\end{cases}\qquad(V.72)$$

i.e. in this example random variable $\omega$ has the uniform distribution law in the gap/interval [-a, a].

Example 3. If

$$r_X(\tau)=e^{-h|\tau|}\cos\beta\tau,$$

then

$$p(\omega)=\frac{h}{\pi}\cdot\frac{\omega^2+h^2+\beta^2}{(\omega^2-2\beta\omega+h^2+\beta^2)(\omega^2+2\beta\omega+h^2+\beta^2)}.\qquad(V.73)$$

Since the integral form of the obtained law of distribution of probabilities takes the form

$$F(z)=\frac{1}{2}+\frac{1}{2\pi}\left[\operatorname{arctg}\frac{z+\beta}{h}+\operatorname{arctg}\frac{z-\beta}{h}\right].\qquad(V.74)$$

we will call this law the law of "arc tangent".

In the particular case with $\beta$=0 is obtained the known law of Cauchy. In a number of cases this law can be replaced with uniform with the parameter

$$\alpha = \frac{\pi}{2} \frac{h^2+\beta^2}{h} q,$$

where

$$0 < q < 1.$$

Quasi-stationary cases. Let us consider the random function of the form

$$Y(t) = b(t) X(t), \qquad (V.75)$$

where X(t) - stationary random function;

b(t) - the preset determined function.

Then

$$M[Y(t)] = m_Y(t) = b(t) M[X(t)] = b(t) m_X(t); \qquad (V.76)$$

$$R_Y(t_1, t_2) = b(t_1) b(t_2) R_X(\tau). \qquad (V.77)$$

Page 148.

From formula (V.75) and (V.77) it follows that the random function

$$\tilde{Y}(t) = b(t)[m_X(t) + V_1 \cos \omega t + V_2 \sin \omega t] \qquad (V.78)$$

will have the same mathematical expectation and correlation function, as function Y(t), if the probabilistic characteristics of the random variables $V_1$, $V_2$ and $\omega$ are selected just as in the stationary case.

Let us consider the more general case. Random function Y(t) is represented in the form

$$Y(t) = \sum_{\kappa=1}^{n} b_\kappa(t) X_\kappa(t), \qquad (V.79)$$
$$\kappa = 1, 2, \ldots, n$$

where $b_\kappa(t)$ — some preset determined functions;

$X_\kappa(t)$ — quasi-stationary random functions, which have mathematical expectations $m_{X_1}(t), m_{X_2}(t), \ldots, m_{X_n}(t)$ and correlation

functions $R_{X_1}(\tau), R_{X_2}(\tau), \ldots, R_{X_n}(\tau)$ respectively.

It is possible to claim that the random function

$$\tilde{Y}(t)=\sum_{\kappa=1}^{n} b_\kappa(t)\,[m_{X_\kappa}(t)+V_{\kappa 1}\cos\omega_\kappa t+V_{\kappa 2}\sin\omega_\kappa t] \qquad (V.80)$$

will have mathematical expectation and correlation function, that coincide identically with the characteristics of random function Y(t), if random variables $V_{\kappa 1}, V_{\kappa 2}, \omega_\kappa$ are independent and their probabilistic characteristics are selected analogously with stationary case examined above.

Actually/really, in this case

$$\left.\begin{aligned} m_Y(t)&=\sum_{\kappa=1}^{n} b_\kappa(t)\,m_{X_\kappa}(t)=m_{\tilde{y}}(t);\\ R_Y(t_1,t_2)&=\sum_{\kappa=1}^{n} b_\kappa(t_1)\,b_\kappa(t_2)\times\\ &\times R_{X_\kappa}(\tau)=R_{\tilde{Y}}(t_1,t_2).\end{aligned}\right\} \qquad (V.81)$$

Formulas (V.81) are valid with n=∞, if infinite series obtained in this case are converging.

Formula (V.79) describes the sufficiently broad class of transient functions, since many transient random processes, which are

encountered during the solution of practical problems, it is possible to represent sufficiently accurately with the help of this formula.

Page 149.

8. Some questions of the theory of interpolation method.

It is shown in work [105] that during the optimum selection of the nodes of interpolation the interpolation method gives the convergent computational processes, if there is a computed probabilistic characteristic, which can be expressed in the form of mathematical expectation from certain function $\chi(V_1, V_2, \ldots, V_m)$.

Let us name its characteristic function of task.

As is known, the mathematical expectation of the function $\chi$ of random variables $V_1, V_2, \ldots, V_m$ is determined in the form of the m-fold integral of the form

$$M[\chi] = \int \ldots \int_{\Delta} \chi(v_1, v_2, \ldots, v_m)\, p(v_1, v_2, \ldots, v_m)\, dv_1 dv_2 \ldots dv_m,$$

where $p(v_1, v_2, \ldots, v_m)$ — density of the joint distribution of the probabilities of the random variables $V_1, V_2, \ldots, V_m$; $\Delta$ - the region of

possible values of the random variables

$$V_1, V_2, \ldots, V_m.$$

Since the cases, which have common sense, this integral there exists, it is possible to speak about the universality of interpolation method during the solution of the problems of accuracy analysis.

The determined selection of nodes and coefficients of interpolation-quadrature formulas is another important property of interpolation method. The random sampling of nodes is realized in the Monte Carlo method, which leads to the dispersion of the obtained results.

It is possible to examine such methods of the accuracy analysis of the nonlinear systems, when not only nodes are random, but also the coefficients of quadrature formulas, i.e., for computing the probabilistic characteristics are applied the formula of the form

$$M[\chi] \approx \sum_{\kappa=1}^{N} A_\kappa \chi(V_{1\kappa}, V_{2\kappa}, \ldots, V_{m\kappa}), \qquad (V.82)$$

where $V_{1\kappa}, V_{2\kappa}, \ldots, V_{m\kappa}$ — values of random variables, obtained in

accordance with certain law $g(v_1, v_2, \ldots, v_m)$, which can differ from the preset density of combined random number distribution $V_1, V_2, \ldots, V_m$; $A_k$ — the coefficients, which are the functions of one or several random variables, which possess certain preset distribution law.

The statistical methods of the accuracy analysis of nonlinear systems and the interpolatio methods, realized during the determined selection of nodes, can be considered as the boundary classes, between which are located the intermediate classes, which possess one or the other special features/peculiarities of the statistical and determined methods.

Page 150.

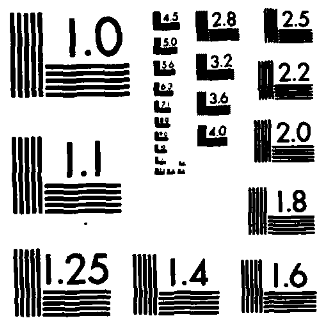
Let us give one of the possible classifications of the methods of the accuracy analysis of nonlinear systems.

1. Versions of values of random variables and values of coefficients of quadrature formula are chosen randomly in accordance with preset laws of distribution of probabilities.

2. Values of coefficients of quadrature formulas are chosen determined, while values of random variables $V_1, V_2, \ldots, V_m$ are chosen random and need not mandatorily coincide with preset density

AD-A143 297 STATIC METHODS IN THE DESIGN OF NONLINEAR AUTOMATIC CONTROL SYSTEMS(U) FOREIGN TECHNOLOGY DIV WRIGHT-PATTERSON AFB OH N I ANDREYEV ET AL. 27 JUN 84 4/9

UNCLASSIFIED FTD-ID(RS)T-1734-83 F/G 12/1 NL




MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

$p(v_1, v_2, \ldots, v_m)$ of their combined doistribution.

3. Values of random variables $V_1, V_2, \ldots, V_m$ are chosen random, and coefficients of quadrature formulas are designed from determined algorithms through obtained values of rando variables.

4. Values of random variables and coefficients of quadrature formulas are determined on determined algorithms. It should be pointed out that the convergent computational algorithms of accuracy analysis exist in each of the classes indicated.

A question about the selection of optimum method among four enumerated classes is complicated problem.

The complexity of this problem consists not only in the variety of different criteria of optimality, but als in the variety of methods.

The algorithms, necessary for the application of interpolation method, were presented in this chapter. Let us note that as the class of the approximating functions in this case was chosen the class of all possible algebraic polynomials of the preset order from the random variables in question.

If we as the approximating functions use other classes of functions, then in each of the classes also it is possible to construct interpolation formula and to solve the task about the construction of optimum interpolation method for the selected class of the approximating functions. In particular, it is possible to use classes of trigonometric polynomials [34], exponential functions, system of the eigenfunctions of different kernels of integral equation, etc. Generally for the construction of interpolation formulas it is possible to use any system of the linearly independent piecewise-continuous functions.

In fact, let ther be system N of the linearly independent functions

$$\psi_1(V),\ \psi_2(V),\ \ldots,\ \psi_N(V).$$

Then function $\chi(V)$ can be approximately represented in the form

$$\chi(V)\approx\sum_{i=1}^{N}C_i\psi_i(V),$$

where $C_i$ — unknown coefficients of expansion.

Page 151.

If we take N nodes $V_1, V_2, \ldots, V_N$ and compute appropriate values

$\chi_1, \chi_2, \ldots, \chi_N$, then by solving system of equations

$$\sum_{s=1}^{N} C_s \psi_s (V_\kappa) = \chi_\kappa, \ \kappa = 1, 2, \ldots, N, \qquad \text{(V.83)}$$

it is possible to determine unambiguously all coefficients $C_s$ according to the formula

$$C_s = \frac{|\Delta_s|}{|\Delta|},$$

where $|\Delta|$ - determinant of matrix/die $\|\psi_s(V_\kappa)\| = \Delta$

(s - number of column,

$\kappa$ - number of line);

$|\Delta_s|$ — definition, obtained from the matrix/die $\Delta$ by the replacement of its column with number s to the column of the values of the right sides of the system of equations (V.83).

Consequently, quadrature formula (V.82) in this case will take this form:

$$M[\chi] \approx \sum_{s=1}^{N} \frac{|\Delta_s|}{|\Delta|} M[\psi_s(V_\kappa)] = \sum_{s=1}^{N} A_s \chi(V_s).$$

It must be noted that nodes $V_\kappa$ in this formula uniquely

determine the values of coefficients $A_n$, moreover these nodes can be chosen randomly or it is determined.

Page 152.

Chapter VI.

# WAYS OF DECREASE OF THE NUMBER OF STATISTICAL NODES DURING THE RESEARCH OF NONLINEAR SYSTEMS.

## 1. General paths.

For the evaluation of the characteristics of automatic control systems taking into account the interaction of random parameters $V_1, V_2, \ldots, V_m$, as has already been spoken, different probabilistic criteria (mathematical expectation and the dispersion of the output coordinate of system, the probability of the nonappearance of coordinate beyond the limits of preset borders, etc.) extensively are used.

On the basis of equality (V.13) by its generalization for arbitrary allocation of values $V_1, V_2, \ldots, V_m$ it is possible to record the expression of criterion of the I evaluation/estimate of the control system in the following form:

$$I=\sum_{\kappa_1,\kappa_2,\ldots,\kappa_m}\Phi_0(t,\mu_{\kappa_1},\mu_{\kappa_2},\ldots,\mu_{\kappa m})\rho_{\kappa_1,\kappa_2,\ldots,\kappa_m}, \qquad (VI.1)$$

where $\mu_{\kappa_1},\mu_{\kappa_2},\ldots,\mu_{\kappa m}$ — values of random variables $V_1, V_2,\ldots, V_m$;

$\rho_{\kappa_1,\kappa_2,\ldots,\kappa_m}$ — the generalized coefficients, computed in this case for the arbitrary law of distribution of values $V_1, V_2,\ldots, V_m$ [in contrast to the formula (V.13)].

Page 153.

Thus, if the generalized function is equal to certain coordinate, i.e.

$$\Phi_0(t,\mu_{\kappa_1},\mu_{\kappa_2},\ldots,\mu_{\kappa m})=y(t,\mu_{\kappa_1},\mu_{\kappa_2},\ldots,\mu_{\kappa m}), \qquad (VI.2)$$

then evaluation/estimate I can take the value of the mathematical expectation of the coordinate in question:

$$I=M[Y(t,V_1,V_2,\ldots,V_m)]\approx$$
$$\approx\sum_{\kappa_1,\kappa_2,\ldots,\kappa_m}y(t,\mu_{\kappa_1},\mu_{\kappa_2},\ldots,\mu_{\kappa m})\rho_{\kappa_1,\kappa_2,\ldots,\kappa_m}, \qquad (VI.3)$$

where $y(t,\mu_{\kappa_1},\mu_{\kappa_2},\ldots,\mu_{\kappa m})$ — value of coordinate Y at the fixed values of random variables $V_1, V_2,\ldots, V_m$;

$\rho_{\kappa_1,\kappa_2,\ldots,\kappa_m}$ — Christoffel number.

Expression (VI.1) is the approximation formula of the numerical determination of the values of the m-fold integral

$$I = \int_{a_1}^{b_1} \dots (m) \dots \int_{a_m}^{b_m} G(V_1, \dots, V_m, t)\, dV_1 \dots dV_m, \quad \text{(VI.4)}$$

of that being the exact value of evaluation/estimate of the I system.

From the selection of the fixed/recorded values

$$\{\mu_{1\kappa_1}, \dots, \mu_{m\kappa_m}\}, \quad \text{(VI.5)}$$

where

$$\kappa_1 = 0, 1, 2, \dots, q_1;\ \kappa_2 = 0, 1, 2, \dots, q_2;$$
$$\dots\dots\dots;\ \kappa_m = 0, 1, 2, \dots, q_m,$$

forming in the m-dimensional space the system of the statistical nodes (this name is caused by the fact that these nodes are located in the region of random variables $V_1, \dots, V_m$), and the selection of generalized coefficients $\rho_{\kappa_1, \kappa_2, \dots, \kappa_m}$ depends substantially the accuracy of the approximate computation of integral (VI.4).

Among a large number of approaches to the selection of statistical nodes (VI.5) and coefficients $\rho_{\kappa_1, \kappa_2, \dots, \kappa_m}$ it is possible to isolate two basic groups of the methods: 1) the methods, which ensure the best approximation of integrand G (or its part) from the point of view of the accuracy of statistical evaluation; 2) the methods, based on the selection of the limited number of statistical nodes and coefficients $\rho_{\kappa_1, \kappa_2, \dots, \kappa_m}$, which ensure acceptable accuracy.

Studies [94] conducted showed that the best approximation of integrand $G(V_1, \dots, V_m, t)$, making it possible to pass to simple formula

(VI.3), provides the system of orthogonal polynomials $H(V_1,..., V_m, t)$.

Page 154.

It is shown in work [105] that if the integrand is represented in the form

$$G(V_1, \ldots, V_m, t) \quad \Phi_0(V_1, \ldots, V_m, t)\prod_{j=1}^{m}\rho_j, \qquad \text{(VI.6)}$$

then, after using the interpolation formula of Lagrange

$$\Phi_0(V_1, \ldots, V_m, t) = \sum_{\kappa_1, \kappa_2, \ldots, \kappa_m} \Phi_0(\mu_{1\kappa_1}, \ldots, \mu_{m\kappa_m}, t) \times$$
$$\times \prod_{j=1}^{m} \frac{H_{j,q+1}(V_j)}{H'_{j,q+1}(\mu_{j\kappa_j})(V_j - \mu_{j\kappa_j})}, \qquad \text{(VI.7)}$$
$$\kappa_1 = 0, 1, 2, \ldots, q_1;\ \kappa_2 = 0, 1, 2, \ldots, q_2;$$
$$\ldots\ldots\ldots\ \kappa_m = 0, 1, 2, \ldots, q_m,$$

where $H_{j,q+1}$ — orthogonal polynomials of degree q+1, it is possible easily to pass from the exact value of evaluation/estimate (VI.4) to expression (VI.1), which gives the exact solution, which coincides with evaluation/estimate (VI.4) for any function $\Phi_0(V_1,..., V_m, t)$, to the described by polynomial degree, which does not exceed 2q+1 [107].

For calculating the evaluation/estimate according to formula (VI.1) with the use/application of an interpolation method, examined in the previous chapter, it is necessary to fulfill

$$N = \prod_{j=1}^{m} (q_j + 1)$$

or

$$N = (q + 1)^m \qquad \text{(VI.8)}$$

with

$$q_1 = q_2 = \ldots = q_m = q$$

the integrations of equations, which describe the system being investigated, substituting each time of the value of statistical nodes (VI.5). With use of Fourier series it is necessary to lead $(2p+1)^m$ the integrations, where p - number of terms of series/row [34].

Meanwhile the integration of the equations of system, even during the use of contemporary high speed TsVM [ЦВМ - digital computer], upon consideration of a considerable number of random disturbances $V_j$ occupies extremely large time. Thus, for m=20 and the low degree of the approximating polynomial q+1=2 a number of necessary integrations $N=2^{20}$ [see formula (VI.8)], which with the expenditure of time for one integration 6 s would require approximately/exemplarily.72 days of computer operation.

Page 155.

The given calculations show that the use/application of methods indicated above with a large number of random disturbances is very difficult. This circumstance led to the creation of other methods,

which make it possible to substantially reduce the number of statistical nodes.

Thus, A. I. Averbukh [1] led further development of the method of equivalent disturbances/perturbations [25] after reducing the number of nonlinear algebraic equations due to the neglect by the members of the third and higher order of smallness. Method makes it possible in this case to be bounded (m+1) to the integration of the equations of system (V.1). But this same-type approach to all m to disturbances/perturbations without taking into account their specific character is not always justified.

A. Ya. Andriyenko [8], examining random variables $V_1, \ldots, V_m$ with the zero mathematical expectations and the moments/torques of odd orders, under certain assumption about allocation of nodes (VI.5) substantially simplifies the system of algebraic equations and reduces a number of nodes, necessary for determining the evaluation/estimate according to formula (V.1). However, the central moments higher than second here are considered in comparison with previous method [1]. One should in this case note that the transition/junction to the random variables with the zero odd moments/torques does not present special difficulties.

A. N. Dobrodeyev [24], being based on assumptions of the

possibility of the approximation of the coordinate of system $Y_i(t, V_1, \dots, V_m)$ being investigated with the m-dimensional polynomial

$$Y_i(V_1, \dots, V_m, t) = H_0 + \sum_{\kappa=1}^{l} \sum_{a_\kappa=\kappa}^{p} \sum_{r_\kappa} \sum_{r_{\kappa-1} \dots r_1} H_{r_1 r_2 \dots r_\kappa}^{(\alpha_1, \alpha_2, \dots, \alpha_\kappa)} \prod_{\nu=1}^{\kappa} V_{r_\nu}^{\alpha_\nu}, \tag{VI.9}$$

where

$$r_\kappa = 1, \dots, m-\kappa+1; \; r_{\kappa-1} = r_\kappa + 1, \dots, m-\kappa; \dots;$$
$$r_1 = r_2 + 1, \dots, m; \; l = \min\{m, p\}; \; a_\kappa = \sum_{\nu=1}^{\kappa} \alpha_\nu,$$

also it comes to the system of the algebraic equations, whose solution makes it possible to determine the value of the moments/torques of the output coordinates of system according to formula (VI.1).

Thus, in all indicated above methods characteristic is the use of the approximating polynomials for replacing exact integral estimation (VI.4) the approximate estimate of form (VI.1).

According to the method with the use of orthogonal polynomials [107] (let us relate it to the 1st group of methods) statistical nodes they are determined by the roots of the selected polynomials, known for the preset law of distribution of independent random quantities $V_1, \dots, V_m$.

Page 156.

Let us note also that during the use of methods of 1st group [107] the nodes of interpolation are chosen from the condition of the minimum of the module of the mathematical expectation of a difference in the generalized characteristic functions

$$R = \left| \int \ldots (m) \ldots \int [\Phi_{0\partial}(V_1, \ldots, V_m, t) - \right.$$
$$\left. - \Phi_0(V_1, \ldots, V_m, t)] \prod_{l=1}^{m} p_l(V_l)\, dV_l \right| \cdot \min, \qquad \text{(VI.10)}$$

where $\Phi_{0\partial}$ — the approximation of function $\Phi_0$.

Four methods [34, 1, 8, 24], which can be attributed to the 2nd group, join single approach to the formation of statistical nodes and coefficients under the method of solution of the additional system of nonlinear algebraic equations. Moreover the selection of nodes (VI.5) and coefficients $\rho_{\kappa_1, \kappa_2, \ldots, \kappa_m}$ is realized from the condition of coincidence with an accuracy to the components/terms/addends of the q order of the smallness of the values of integral (VI.4) after the approximate approximation of function $G(V_1, \ldots, V_m, t)$ and formula

$$I = \sum_{\kappa_1, \ldots, \kappa_m} G(\mu_{1\kappa_1}, \ldots, \mu_{m\kappa_m}, t)\, \rho_{\kappa_1, \ldots \kappa_m}, \qquad \text{(VI.11)}$$

where the function is decomposed in the Maclaurin series.

If we take into account also the circumstance that for the approximation of integrand, described by the polynomial of the q order, according to the method of the 2nd group the expansion of the same q order is required, and for the methods of the 1st group it

suffices to take the polynomial, whose order $q_2=(q-1)/2$, then it is obvious that with the equal degree of the approximating polynomial the 1st group of methods gives high degree of accuracy. However, in this case a larger number of interpolation (statistical) nodes will be required.

The advantage of the methods of the 2nd group consists, first of all, of the considerable decrease of a number of statistical nodes, however, as it is noted in work [34], with a large number of disturbances/perturbations m for determining the statistical evaluations of system are required hundreds of integrations of reference system. The methods, proposed in works [1, 8], after some transformations of the reference system of disturbances/perturbations make it possible to considerably reduce the number of disturbances/perturbations. However, the 2nd group of methods requires the carrying out of sufficiently complicated pretreatments (solution of the system of further nonlinear algebraic equations, construction of matrices [8] and some others). At the same time in the 1st group it suffices to switch over to the standard distribution laws, to select the roots of orthogonal polynomials from tables [107], to lead the solution of reference system, also, by simple formula (VI.1) to find the value of statistical characteristics.

Page 157.

The single form of formula for determining the statistical evaluation becomes especially important during the calculation of the probability that it is difficult in the 2nd group of methods. Thus, for instance, for determining the probability of falling of coordinate Y in region with a radius of C formula (VI.1) will take the form

$$P[Y \leqslant C] \approx \sum_{\kappa_1, \ldots, \kappa_m} X_{\kappa_1, \ldots, \kappa_m} \rho_{\kappa_1, \ldots, \kappa_m}, \qquad \text{(VI.12)}$$

where $X_{\kappa_1, \ldots, \kappa_m}$ — characteristic function, equal to 1 with $Y_{\kappa_1, \ldots, \kappa_m} \leqslant C$ and 0 when $Y_{\kappa_1, \ldots, \kappa_m} > C$.

All the enumerated circumstances lead to the need for further improvement of the indicated methods of the study (analysis and synthesis) of the complex multichannel nonlinear systems of control mainly by the decrease of a number of statistical nodes, necessary for determining the evaluation/estimate of system. Among the series/row of the directions, which decide stated problem, let us pause only at two: the use of the minimum approximating polynomials with the determination of the evaluation/estimate of system by determining the necessary coefficients of expansion; the decrease of a number of statistical nodes by transition/junction through the equations of relation from the preset m-dimensional region of the

random variables to the region of their substantially smaller number.

During the use of a method of the approximating polynomials a number of statistical nodes, which determine the number of required integrations, can be varied, satisfying the condition

$$\prod_{i=1}^{m}(q_i+1)\leqslant N. \qquad \text{(VI.13)}$$

With the approximation of integrand $G(V_1,\ldots,V_m,t)$ with Maclaurin series or by any other polynomial it is possible to use expansion with the different degree of polynomials in terms of random variables $V_1,\ldots,V_m$, for example

$$\begin{aligned} G(V_1,\ldots,V_m,t) &= G_0+\frac{\partial G}{\partial V_1}V_1+ \\ &+\frac{\partial G}{\partial V_2}V_2+\ldots+\frac{\partial G}{\partial V_m}V_m+\frac{\partial^2 G}{\partial V_{m-2}^2}V_{m-2}^2+ \\ &+\frac{\partial^2 G}{\partial V_{m-1}^2}V_{m-1}^2+\frac{\partial^2 G}{\partial V_m^2}V_m^2. \end{aligned} \qquad \text{(VI.14)}$$

Finally, it is possible to use a combination of methods, i.e., for obtaining the preliminary evaluations/estimates of system (especially with synthesis) to use one of the methods indicated, which requires a smaller number of integrations of the equations of system, and for the final evaluation/estimate to use more exact method with a large number of integrations.

Page 158.

2. Method of minimum polynomials.

One of the fundamental difficulties, as it was indicated above, are the approximation of integrand $G(t, V_1, \ldots, V_m)$, the ensuring required accuracy of the statistical evaluation of system. In order to exclude error, it is possible to select the previously high degree of the approximating polynomial. However, with a large number of random variables this path proves to be unacceptable due to an enormous number of integrations of equations of system. Therefore another path more frequently is chosen: after assigning certain degree of the approximating polynomial, is obtained evaluation/estimate $T_1$ of system, and further, raising the degree of polynomial, is designed the second value of $T_2$ of evaluation/estimate also according to some inequalities, for example

$$|I_1 - I_2| \leqslant \delta_1, \qquad \text{(VI. 15)}$$

where $\delta_1$ - allowable absolute error in the evaluation/estimate, or

$$\left|\frac{I_1 - I_2}{I_1}\right| \leqslant \delta_2, \qquad \text{(VI. 16)}$$

where $\delta_2$ - allowable relative error in the evaluation/estimate, they check the accuracy of the calculation of the value of evaluation/estimate.

With the second method of obtaining the reliable value of

evaluation/estimate, especially with a large number of random disturbances, applied to the system, it is possible to assign n+1 statistical nodes (VI.5) in the m-dimensional region of random variables. Moreover must be made the equality

$$\prod_{l=1}^{m}(q_l+1)=n+1. \quad \text{(VI.17)}$$

From these n+1 nodes it is necessary to calculate the evaluation/estimate of system. In the given case is legal the formulation of the problem about finding of minimum polynomial $G(t, V_1, \ldots, V_m)$ (the lowest possible degree), which in selected nodes (VI.5) takes the values

$$G(t, \mu_{1\kappa_1}, \ldots, \mu_{m\kappa_m}), \quad \text{(VI.18)}$$
$$\kappa_1=0, 1, 2, \ldots, q_1;\ \kappa_2=0, 1, 2, \ldots, q_2;$$
$$\ldots \ldots \ldots \kappa_m=0, 1, 2, \ldots, q_m,$$

obtained on the base of the integration of system of equations, which describe the dynamics of the system of control (whence ensues/escapes/flows out name - method of minimum polynomials).

Page 159.

Let us pause briefly at the mathematical formalization of the task in question. Let polynomial $G(t, V_1, \ldots, V_m)$ take the form

$$G(t, V_1, \ldots, V_m)=a_0+\sum_{\nu=1}^{m}\sum_{\kappa_1=1}^{q_1}\sum_{\kappa_2=1}^{q_2}\ldots\sum_{\kappa_\nu=1}^{q_\nu}c_{\kappa_1\kappa_2\ldots\kappa_\nu}V_{\kappa_1}V_{\kappa_2}\ldots V_{\kappa_\nu}. \quad \text{(VI.19)}$$

If we superimpose on polynomial $G(t, V_1, \dots, V_m)$ of no further conditions, then, obviously, its degree on all disturbances/perturbations must be such that a number of unknown coefficients $a_0, a_{\kappa_1\kappa_2\dots\kappa_m}$ $(\kappa_1 = 1, 2, \dots, q_1;\ \kappa_2 = 1, 2, \dots, q_2; \dots;\ \kappa_m = 1, 2, \dots, q_m)$ would be equal to a number of statistical nodes n+1. Actually/really, after taking n+1 sample from the system of nodes (VI.5) and after determining n+1 the value of function (VI.18), after their substitution into equality (VI.19) we will obtain the linear system of equations, for example,

$$\left.\begin{array}{c}
G(t, \mu_{11}, \dots, \mu_{m1}) = c_0 + \\
+\sum\limits_{\nu=1}^{m}\sum\limits_{\kappa_1=1}^{q_1}\sum\limits_{\kappa_2=1}^{q_2}\dots\sum\limits_{\kappa_\nu=1}^{q_\nu} c_{\kappa_1\kappa_2\dots\kappa_\nu}\mu_{\kappa_1 1}\dots\mu_{\kappa_\nu 1}; \\
G(t, \mu_{12}, \dots, \mu_{m2}) = a_0 + \\
+\sum\limits_{\nu=1}^{m}\sum\limits_{\kappa_1=1}^{q_1}\sum\limits_{\kappa_2=1}^{q_2}\dots\sum\limits_{\kappa_\nu=1}^{q_\nu} a_{\kappa_1\kappa_2\dots\kappa_\nu}\underset{(12)}{\mu_{\kappa_1 2}}\dots\underset{(m2)}{\mu_{\kappa_\nu 2}}; \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
G(t, \mu_{1q_1}, \dots, \mu_{mq_m}) = a_0 + \\
+\sum\limits_{\nu=1}^{m}\sum\limits_{\kappa_1=1}^{q_1}\sum\limits_{\kappa_2=1}^{q_2}\dots\sum\limits_{\kappa_\nu=1}^{q_\nu} c_{\kappa_1\kappa_2\dots\kappa_\nu}\underset{(1q_1)}{\mu_{1\kappa_1}}\dots\underset{(\nu q_\nu)}{\mu_{\nu\kappa_\nu}},
\end{array}\right\} \quad (VI.20)$$

solution of which does not cause fundamental difficulties.

After the determination of the coefficients of the polynomial (VI.19) the value of statistical evaluation, for example the correlation function of any order, can be determined according to the expression

$$M\begin{bmatrix} Y_{i_1}^{p_1}, Y_{i_2}^{p_2}, \dots, Y_{i_n}^{p_n} \\ t_1, t_2, \dots, t_n \end{bmatrix} = \int_{v_1}^{v_1'} \dots (m) \dots \int_{v_m}^{v_m'} G(V_1, \dots, V_m, t) \times$$
$$\times p(V_1, \dots, V_m)\, dV_1 \dots dV_m, \qquad \text{(VI.21)}$$
$$p_1 = 0, 1, 2, \dots;\ p_2 = 0, 1, 2, \dots;\ \dots\dots;\ p_n = 0, 1, 2, \dots;$$
$$i_1 = 1, 2, \dots, n;\ i_2 = 1, 2, \dots, n; \dots\dots;\ i_n = 1, 2, \dots, n.$$

Page 160.

For this let us substitute into equality (VI.21) approximate value of function $G(t, V_1, \dots, V_m)$:

$$M\begin{bmatrix} Y_{i_1}^{p_1}, Y_{i_2}^{p_2}, \dots, Y_{i_n}^{p_n} \\ t_1, t_2, \dots, t_n \end{bmatrix} \approx \int_{v_1}^{v_1'} \dots (m) \dots \int_{v_m}^{v_m'} \Big[ a_0 +$$
$$+ \sum_{\nu=1}^{m} \sum_{\kappa_1=1}^{q_1} \sum_{\kappa_2=1}^{q_2} \dots \sum_{\kappa_\nu=1}^{q_\nu} a_{\kappa_1 \kappa_2 \dots \kappa_\nu} V_{\kappa_1} V_{\kappa_2} \dots V_{\kappa_\nu} \Big] \times$$
$$\times p(V_1, V_2, \dots, V_m)\, dV_1 dV_2 \dots dV_m, \qquad \text{(VI.22)}$$

or, resetting the operations of addition,

$$M\begin{bmatrix} Y_{i_1}^{p_1}, Y_{i_2}^{p_2}, \dots, Y_{i_n}^{p_n} \\ t_1, t_2, \dots, t_n \end{bmatrix} = a_0 +$$
$$+ \sum_{\nu=1}^{m} \sum_{\kappa_1=1}^{q_1} \sum_{\kappa_2=1}^{q_2} \dots \sum_{\kappa_\nu=1}^{q_\nu} a_{\kappa_1 \kappa_2 \dots \kappa_\nu} \int_{v_1}^{v_1'} \dots (m) \dots \int_{v_m}^{v_m'} V_{\kappa_1} \dots V_{\kappa_\nu} \times$$
$$\times p(V_1, \dots, V_m)\, dV_1 \dots dV_m, \qquad \text{(VI.23)}$$

since

$$\int_{v_1}^{v_1'} \dots (m) \dots \int_{v_m}^{v_m'} a_0 p(V_1, \dots, V_m)\, dV_1 \dots dV_m = a_0.$$

During the deployment of repeated sum in expression (VI.23) the integrals of the form

$$\int\limits_{v_1}^{v_1'} \ldots (m) \ldots \int\limits_{v_m}^{v_m'} \prod_{j=1}^{r} V_{\kappa_j} p(V_1, \ldots, V_m)\, dV_1 \ldots dV_m = \mathrm{M}\left[\prod_{j=1}^{r} V_{\kappa_j}\right], \tag{VI.24}$$

$$r = 1, 2, \ldots, m;\ \kappa_1 = 1, 2, \ldots, q_1;$$
$$\kappa_2 = 1, 2, \ldots, q_2; \ldots\ldots;\ \kappa_m = 1, 2, \ldots, q_m.$$

will be obtained.

Page 161.

Expression (VI.24) substantially is simplified for independent random quantities $V_1, \ldots, V_m$. In this case the density of the distribution

$$p(V_1, \ldots, V_m) = \prod_{j=1}^{m} p(V_j), \tag{VI.25}$$

but repeated integral (VI.24) can be represented as the product of the integrals of the form

$$\int\limits_{v_j}^{v_j'} V_j^{2l} p(V_j)\, dV_j = \mathrm{M}\left[V_j^{2l}\right], \tag{VI.26}$$

$$l = 1, 2, \ldots, \frac{m}{2}\left[-\frac{1}{2}\right],\ j = 1, 2, \ldots, m;$$

$$\int\limits_{v_j}^{v_j'} V_j^{2l+1} p(V_j)\, dV_j = \mathrm{M}\left[V_j^{2l+1}\right], \tag{VI.27}$$

$$l = 0, 1, 2, \ldots, \frac{m}{2} - 1\left[-\frac{1}{2}\right];$$

$$\int\limits_{v_j}^{v_j'} p(V_j)\, dV_j = 1, \tag{VI.28}$$

$$j = 1, 2, \ldots, m.$$

If the densities of distribution $p(V_j)$ — even functions, then

integrals (VI.27) are equal to zero, i.e.

$$\int_{v_j}^{v_j'} V_j^{2l+1} p(V_j)\,dV_j = 0, \qquad \text{(VI.29)}$$

since product of odd function $V_j^{2l+1}$ to even $p(V_j)$ gives odd function, and limits $v_j$ and $v_j'$ have the equal modules

$$|v_j| = |v_j'| = |-v_j|. \qquad \text{(VI.30)}$$

Using the obtained values of integrals (VI.26) (VI.27) (VI.28), expression (VI.23) in general form can be converted to the form

$$M\begin{bmatrix} Y_{t_1}^{p_1}, & Y_{t_2}^{p_2}, & \dots, & Y_{t_n}^{p_n} \\ t_1, & t_2, & \dots, & t_n \end{bmatrix} \approx c_0 + \\ + \sum_{\nu=1}^{m} \sum_{\varkappa_1=1}^{q_1} \sum_{\varkappa_2=1}^{q_2} \cdots \sum_{\varkappa_\nu=1}^{q_\nu} a_{\varkappa_1\varkappa_2\dots\varkappa_\nu} M[V_{\varkappa_1}V_{\varkappa_2}\dots V_{\varkappa_\nu}] \qquad \text{(VI.31)}$$

and for the independent random quantities with the even density function of allocation

$$M\begin{bmatrix} Y_{t_1}^{p_1}, & Y_{t_2}^{p_2}, & \dots, & Y_{t_n}^{p_n} \\ t_1, & t_2, & \dots, & t_n \end{bmatrix} \approx a_0 + \\ + \sum_{\nu=1}^{m} \sum_{\varkappa_1=1}^{q_1} \sum_{\varkappa_2=1}^{q_2} \cdots \sum_{\varkappa_\nu=1}^{q_\nu} a_{\mathring{q}_i\mathring{q}_j\dots\mathring{q}_s} M\left[V_i^{\mathring{q}_i} V_j^{\mathring{q}_j} \dots V_s^{\mathring{q}_s}\right], \qquad \text{(VI.32)}$$

where $\mathring{q}_i, \mathring{q}_j, \dots, \mathring{q}_s$ — even degrees, determined by the even degrees of the approximating polynomial.

Page 162.

It must be noted that the transition/junction to the independent random quantities with the even density function of allocation does not present fundamental difficulties. Therefore obtained formula

(VI.32) can be accepted as fundamental, since a number of required coefficients $a_0, a_{q_i^* q_j^* \dots q_s^*}$ in it is substantially less than coefficients $a_0, a_{\kappa_1 \kappa_2 \dots \kappa_m}$ in formula (VI.32).

Thus, for determining the moment/torque of any order of the output coordinates of system it is necessary to lead n+1 the integration of reference system, to determine coefficients $a_{q_i^* q_j^* \dots q_s^*}$ from the linear system of equations (VI.20) and to use simple formula (VI.32).

It remains, however, thus far not solved a question about the selection of statistical nodes. For their determination let us represent approximating polynomial (VI.19) in other form, for which we will use auxiliary polynomial $w(V_1, \dots, V_m)$, which satisfies the conditions

$$w(V_1, \dots, V_m) = \begin{cases} 0 \ \overset{(1)}{\text{при}} \ V_j \neq \mu_{j\kappa_j}, \ j = 1, 2, \dots, m, \\ 1 \ \text{при} \ V_j = \mu_{j\kappa_j}, \ j = 1, 2, \dots, m. \end{cases} \quad \text{(VI.33)}$$

Key: (1). with.

Such polynomial can be the polynomial of the effect of nodes $\mu_{j\kappa_j}$ $j = 1, 2, \dots, m$, which is written/recorded in the form

$$w(V_1, \dots, V_m) = \\ = \prod_{j=1}^{m} \frac{(V_j - \mu_{1\kappa_1}) \cdots (V_j - \mu_{j-1\,\kappa_{j-1}})(V_j - \mu_{j+1\,\kappa_{j+1}}) \cdots (V_j - \mu_{m\kappa_m})}{(\mu_{j\kappa_j} - \mu_{1\kappa_1}) \cdots (\mu_{j\kappa_j} - \mu_{j-1\,\kappa_{j-1}})(\mu_{j\kappa_j} - \mu_{j+1\,\kappa_{j+1}}) \cdots (\mu_{j\kappa_j} - \mu_{m\kappa_m})}, \quad \text{(VI.34)}$$

or

$$w(V_1, \ldots, V_m) = \prod_{j=1}^{m} \frac{w_{j,q_j+1}(V_j)}{w'_{j,q_j+1}(\mu_{j\kappa_j})(V_j - \mu_{j\kappa_j})}, \quad \text{(VI.35)}$$

where

$$w_{j,q_j+1}(V_j) = (V_j - \mu_{1\kappa_j})(V_j - \mu_{2\kappa_j}) \ldots (V_j - \mu_{m\kappa_j}); \quad \text{(VI.36)}$$

$w'_{j,q_j+1}(\mu_{j\kappa_j})$ — derivative of $w_{j,q_j+1}(V_j)$ on $V_j$, calculated at point $V_j = \mu_{j\kappa_j}$.

Page 163.

Then approximating polynomial (VI.19) can be recorded in the form of the interpolation formula of Lagrange:

$$G(t, V_1, \ldots, V_m) = \sum_{\kappa_1, \ldots, \kappa_m} G(t, \mu_{1\kappa_1}, \ldots, \mu_{m\kappa_m}) \times$$
$$\times \prod_{j=1}^{m} \frac{w_{j,q_j+1}(V_j)}{w'_{j,q_j+1}(\mu_{j\kappa_j})(V_j - \mu_{j\kappa_j})}, \quad \text{(VI.37)}$$
$$\kappa_1 = 0, 1, 2, \ldots, q_1;\ \kappa_2 = 0, 1, 2, \ldots, q_2;$$
$$\ldots\ldots\ldots\ \kappa_m = 0, 1, 2, \ldots, q_m,$$

i.e.

$$a_0 + \sum_{\nu=1}^{m} \sum_{\kappa_1=1}^{q_1} \sum_{\kappa_2=1}^{q_2} \cdots \sum_{\kappa_\nu=1}^{q_\nu} a_{\kappa_1\kappa_2\ldots\kappa_\nu}(t) V_{\kappa_1} V_{\kappa_2} \ldots V_{\kappa_\nu} =$$
$$= \sum_{\kappa_1, \ldots, \kappa_m} G(\mu_{1\kappa_1}, \ldots, \mu_{m\kappa_m}, t) \cdot \prod_{j=1}^{m} \frac{w_{j,q_j+1}(V_j)}{w'_{j,q_j+1}(\mu_{j\kappa_j})(V_j - \mu_{j\kappa_j})}. \quad \text{(VI.38)}$$

The right side of equality (VI.38) makes it possible to obtain $\prod_{l=1}^{m}(q_l+1)$ statistical nodes (VI.5). But since a number of unknown coefficients on the left side of equality (VI.38) is considerably less, then it suffices from entire set (VI.5) to select such, which would provide the necessary condition of solving algebraic system (VI.20), i.e., it is necessary to select such set of nodes so that the determinant of systems (VI.20) would satisfy Vandermondes's condition

$$W(\mu_{1\kappa_1},\ldots,\mu_{m\kappa_m})=\begin{vmatrix}1 & C_1^1 & C_1^2 & \ldots & C_1^s\\ 1 & C_2^1 & C_2^2 & \ldots & C_2^s\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & C_s^1 & C_s^2 & \cdots & C_s^s\end{vmatrix}, \qquad \text{(VI.39)}$$

where $C_l^j$ — calculated values of values $\prod_{l=1}^{v}\mu_{l\kappa_l}$, which belong to coefficients $c_{\kappa_1\kappa_2\ldots\kappa_m}$.

Page 164.

If we substitute into expression (VI.38) for polynomials $w_{l,q_l+1}(V_l)$ orthogonal polynomials $H_{l,q_l+1}(V_l)$, which ensure the best approximation of the function being investigated, then, obviously, as statistical nodes (VI.5) can be selected Chebyshev's nodes. Consequently, using the precomputed roots, according to prime formulas [107] is possible sufficiently simply to determine the values of statistical nodes (VI.5). For example, for the normal law

of distribution p(v) these values

$$V_l^{(q_l+1)} = m_{v_l} + \sigma_{v_l} \sqrt{2} x_l^{(q_l+1)}, \qquad \text{(VI.40)}$$

where $x_l^{(q_l+1)}$ — roots of orthogonal polynomial $H_{l,q_l+1}$;

$m_{v_l}$ — the mathematical expectation of random variable $V_j$;

$\sigma_{v_l}$ — standard deviation of random variable $V_j$.

For the exponential law of distribution $p(v) = p_0 e^{-\gamma v}$, $\gamma > 0$

$$V_l^{(q_l+1)} = \frac{1}{\gamma} x_l^{(q_l+1)}. \qquad \text{(VI.41)}$$

For the uniform law of distribution $p(v) = \frac{1}{v_l - v_l'}$

$$V_l^{(q_l+1)} = \frac{v_l - v_l'}{2} x_l^{(q_l+1)} + \frac{v_l + v_l'}{2}. \qquad \text{(VI.42)}$$

The method stated above of the statistical evaluation of system is in certain kind the combined method, since it is very simply reduced to the methods of the 1st and 2nd groups. A number of required statistical nodes here lies/rests over a wide range from 1 to $(2q+1)^m$.

Actually/really, let us represent polynomial $G(V_1, \ldots, V_m, t)$ in the form

$$G(V_1, \ldots, V_m, t) = Y(V_1, \ldots, V_m, t) \approx$$
$$\approx a_0(t) + a_1(t) V_1 + a_2(t) V_2 + \ldots + a_m(t) V_m,$$

where $Y(V_1, \ldots, V_m, t)$ — the coordinate of system being investigated, which is the function of independent random quantities with the even density function of allocation.

Then for determining the mathematical expectation of coordinate $Y(V_1, \ldots, V_m, t)$ it suffices to know only the value of coefficient of $a_0$, i.e.,

$$M[Y(V_1, \ldots\ldots, V_m, t)] = \int_{v_1}^{v_1'} \ldots (m) \ldots \int_{v_m}^{v_m'} \sum_{l=0}^{m} a_l V_l \times$$
$$\times \prod_{l=1}^{m} p(V_l)\, dV_l = a_0. \qquad \text{(VI.43)}$$

Page 165.

Consequently, after assigning values of random variables for statistical node $\{\mu_{11}, \ldots, \mu_{mm}\}$, it is possible to find the value of coefficient of $a_0$ when

$$a_1\mu_{11} + a_2\mu_{22} + \ldots + a_m\mu_{mm} = 0. \qquad \text{(VI.44)}$$

This approach superimposes certain uncertainty/indeterminacy on the determination of coefficients $a_1, a_2, \ldots, a_m$ and, therefore, to the accuracy of calculations. The given equation for determining the coefficients $a_1, a_2, \ldots, a_m$ can be decomposed into the series/row of the

equations, which form the subsystem

$$\left.\begin{aligned} a_1\mu_{11} + a_2\mu_{22} + \ldots + a_l\mu_{ll} = 0; \\ a_{l+1}\mu_{l+1,\,l+1} + a_{l+2}\mu_{l+2,\,l+2} + \ldots + a_j\mu_{jj} = 0; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_s\mu_{ss} + a_{s+1}\mu_{s+1,\,s+1} + \ldots + a_m\mu_{mm} = 0. \end{aligned}\right\} \quad \text{(VI.45)}$$

However, this will influence an error in the determination of statistical evaluation, although will simplify the determination of coefficients $a_1, a_2, \ldots, a_m$. It is appropriate to note that the determination of the values of these coefficients for determination $\mu[Y(V_1, \ldots, V_m, t)]$ is not required, but has only that sense, that approximating function $G(t, V_1, \ldots, V_m)$ must be more exact.

Obtaining the additional solutions of system n+1<m+1, it is possible to raise degree of accuracy. In order to compute coefficient of $a_0$ at the values of coefficients $a_1 \neq 0, a_2 \neq 0, \ldots, a_m \neq 0$, found from condition $Y(t, V_1, \ldots, V_m) = Y(t, \mu_{1\varkappa_1}, \ldots, \mu_{m\varkappa_m})$, necessary to have m+1 a solution of system, i.e., m+1 the value of coordinate $Y(t, \mu_{1\varkappa_1}, \ldots, \mu_{m\varkappa_m})$. The accuracy of calculation in this case grows.

With an increase in the approximating polynomial a number of required solutions will, naturally, increase; however, by the selection of the values of coefficients $c_{\varkappa_1\varkappa_2\ldots\varkappa_m}$ it it is possible to decrease, using the following method.

Since for determining the statistical evaluation according to formula (VI.33) it suffices to know only coefficients $a_{\dot{q}_i\dot{q}_j\ldots\dot{q}_s}$ from general/common/total number of all coefficients $a_{\kappa_1\kappa_2\ldots\kappa_m}$, we will choose the remaining coefficients of system (VI.20) thus.

Page 166.

Let us represent algebraic system (VI.20) in the form

$$\left.\begin{aligned} G(\mu_{11},\ldots,\mu_{m1},t) &= a_0+\Big(\sum_{\dot{q}_i}\sum_{\dot{q}_j}\cdots\sum_{\dot{q}_s}a_{\dot{q}_i\dot{q}_j\ldots\dot{q}_s}+\\ &+\sum_{p_i}\sum_{p_j}\cdots\sum_{p_s}a_{p_ip_j\ldots p_s}\Big)\mu_{\underset{(11)}{1\kappa_1}}\cdots\mu_{\underset{(m1)}{m\kappa_m}};\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ G(\mu_{1q_1},\ldots\mu_{mq_m},t) &= a_0+\Big(\sum_{\dot{q}_i}\sum_{\dot{q}_j}\sum_{\dot{q}_s}a_{\dot{q}_i\dot{q}_j\ldots\dot{q}_s}+\\ &+\sum_{p_j}\sum_{p_j}\cdots\sum_{p_s}a_{p_ip_j\ldots p_s}\Big)\mu_{\underset{(1q_1)}{1\kappa_1}}\cdots\mu_{\underset{(mq_m)}{m\kappa_m}};\end{aligned}\right\}\quad\text{(VI.46)}$$

let us select coefficients $a_{p_ip_j\ldots p_s}$ solving the system

$$\left.\begin{aligned} &\sum_{p_i}\sum_{p_j}\cdots\sum_{p_s}a_{p_ip_j\ldots p_s}\mu_{\underset{(11)}{1\kappa_1}}\cdots\mu_{\underset{(m1)}{m\kappa_m}}=0;\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &\sum_{p_i}\sum_{p_j}\cdots\sum_{p_s}a_{p_ip_j\ldots p_s}\mu_{\underset{(1q_1)}{1\kappa_1}}\cdots\mu_{\underset{(mq_m)}{m\kappa_m}}=0.\end{aligned}\right\}\quad\text{(VI.47)}$$

Then system (VI.46) will take the form

$$\left.\begin{array}{l} G(\mu_{11}, \ldots, \mu_{m1}, t) = a_0 + \sum\limits_{\dot{q}_l} \sum\limits_{\dot{q}_j} \cdots \sum\limits_{\dot{q}_s} a_{\dot{q}_l \dot{q}_j \ldots \dot{q}_s} \times \\ \quad \times \mu_{\underset{(11)}{1\kappa_1}} \cdots \mu_{\underset{(m1)}{m\kappa_m}}; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ G(\mu_{1q_1}, \ldots, \mu_{mq_m}, t) = a_0 + \sum\limits_{\dot{q}_l} \sum\limits_{\dot{q}_j} \cdots \sum\limits_{\dot{q}_s} a_{\dot{q}_l \dot{q}_j \ldots \dot{q}_s} \times \\ \quad \times \mu_{\underset{(1q_1)}{1\kappa_1}} \cdots \mu_{\underset{(mq_m)}{m\kappa_m}}. \end{array}\right\} \quad \text{(VI.48)}$$

The values of coefficients $a_{p_l p_j \ldots p_s}$ in this case in the general case are not equal to zero, but their effect on the calculation of coefficients $a_{\dot{q}_l \dot{q}_j \ldots \dot{q}_s}$ is excluded. If some of equations (VI.47) are incompatible, then it is necessary to attach to the part of coefficients $a_{p_l p_j \ldots p_s}$ the values, equal to zero. For example, let

$$G(V_1, \ldots, V_m, t) = Y(t_1, V_1, V_2) = a_0 + a_1V_1 + a_2V_2 + \\ + a_{12}V_1V_2 + a_{11}V_1^2 + a_{22}V_2^2, \quad \text{(VI.49)}$$

but the mathematical expectation of coordinate $Y(t, V_1, V_2)$

$$M[Y(t, V_1, V_2)] = a_0 + a_{11}M[V_1^2] + a_{22}M[V_2^2], \quad \text{(VI.50)}$$

i.e. be sufficient to have values of three coefficients of $a_0$, $a_{11}$, $a_{22}$.

Page 167.

Let us compose the system of three algebraic equations for their determination:

$$\left.\begin{aligned} Y(t_1, \mu_{11}, \mu_{21}) &= a_0 + a_1\mu_{11} + a_2\mu_{21} + a_{12}\mu_{11}\mu_{21} + \\ &+ a_{11}\mu_{11}^2 + a_{22}\mu_{21}^2; \\ Y(t_1, \mu_{12}, \mu_{22}) &= a_0 + a_1\mu_{12} + a_2\mu_{22} + a_{12}\mu_{12}\mu_{22} + \\ &+ a_{11}\mu_{12}^2 + a_{22}\mu_{22}^2; \\ Y(t_1, \mu_{13}, \mu_{23}) &= a_0 + a_1\mu_{13} + a_2\mu_{23} + a_{12}\mu_{13}\mu_{23} + \\ &+ a_{11}\mu_{13}^2 + a_{22}\mu_{23}^2, \end{aligned}\right\} \quad (VI.51)$$

where coefficients $a_1$, $a_2$, $a_{12}$ can be determined from the conditions

$$\left.\begin{aligned} a_1\mu_{11} + a_2\mu_{21} + a_{12}\mu_{11}\mu_{21} &= 0; \\ a_1\mu_{12} + a_2\mu_{22} + a_{12}\mu_{12}\mu_{22} &= 0; \\ a_1\mu_{13} + a_2\mu_{23} + a_{12}\mu_{13}\mu_{23} &= 0. \end{aligned}\right\} \quad (VI.52)$$

Then we find the required coefficients $a_0$, $a_{11}$, $a_{22}$ from the system

$$\left.\begin{aligned} Y(t_1, \mu_{11}, \mu_{21}) &= a_0 + a_{11}\mu_{11}^2 + a_{22}\mu_{21}^2; \\ Y(t_1, \mu_{12}, \mu_{22}) &= a_0 + a_{11}\mu_{12}^2 + a_{22}\mu_{22}^2; \\ Y(t_1, \mu_{13}, \mu_{23}) &= a_0 + a_{11}\mu_{13}^2 + a_{22}\mu_{23}^2. \end{aligned}\right\} \quad (VI.53)$$

Again one should stress that systems (VI.47) and (VI.52) are solved not for obtaining the evaluation/estimate of system, but only when for some other targets it is necessary to obtain all values of coefficients $a_{\kappa_1\kappa_2\ldots\kappa_m}$. It is at the same time obvious that coefficients $a_{p_i p_j \ldots p_s}$ affect indirectly the accuracy of the determination of coefficients $a_0, a_{\dot{q}_i \dot{q}_j \ldots \dot{q}_s}$, accuracy of the value of the evaluation/estimate of system.

Thus, the method of determining the statistical evaluation of system presented uses a series/row of the advantages of the methods

of the 1st and 2nd groups and it is in a sense intermediate. This method does not require the solution of the nonlinear system of algebraic equations for determining the statistical nodes, since the nodes, calculated for the methods of the 1st group, are used, But the coefficients of the approximating (interpolation) polynomial are determined by solution of linear system of equations.

The necessary number of statistical nodes lies/rests in the range from 1 to $(2q+1)^m$. The increase in the number of nodes raising the accuracy of calculations.

Page 168.

But, as the studies of the series/row of systems [1, 8, 24, 34], selection of a number of nodes N in the range

$$m+1 \leqslant N \leqslant 2^m, \qquad \text{(VI.54)}$$

where m - number of random disturbances, are shown, it provides the high accuracy of the determination of the statistical evaluation of the nonlinear control systems.

Example. For the illustration of the method presented let us consider simple system [34], described by equation $(dY/dt)=\sqrt{1-V^2Y^2}$, with one random parameter V (Fig. VI.1), subordinated to the law of uniform probability density in the segment [-(1/2), +( 1/2)]. In this

case mathematical expectation values the V and initial moment/torque of the third order M [V³] are equal to zero, i.e.

$$m_V = M[V^3] = 0.$$

The initial moment of the second order can be determined from the expression

$$M[V^2] = \int_{-\frac{1}{2}}^{+\frac{1}{2}} (V + m_V)^2\, dV = \frac{1}{12}, \qquad \text{(VI.55)}$$

but the value of the initial moment of the fourth order is determined by the equality

$$M[V^4] = \int_{-\frac{1}{2}}^{+\frac{1}{2}} (V + m_V)^4\, dV = \frac{1}{80}. \qquad \text{(VI.56)}$$

Exact expression for the coordinate of system with Y(0)=0 takes the form

$$Y = \frac{\sin Vt}{V}. \qquad \text{(VI.57)}$$

Let us find mathematical expectation $m_Y$ of coordinate Y of system for different moments/torques of time t employing procedure presented above and let us compare it with the exact value:

$$M[Y] = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\sin Vt}{V}\, dV = 2\,\mathrm{Si}\left(\frac{t}{2}\right). \qquad \text{(VI.58)}$$

Let us assume that for this function Y(t, V) can be approximated

for the prescribed moment/torque of time $t_1$ by the expression

$$Y(t_1 V) = \sum_{i=0}^{5} a_i V^i.$$

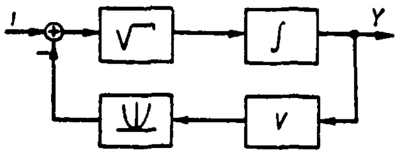



Fig. VI.1. Example of the structural scheme of nonlinear system.

Page 169.

Then the approximation formula of the unknown evaluation/estimate of system will take the form

$$\tilde{M}(t) \approx \int_{-\frac{1}{2}}^{+\frac{1}{2}} \sum_{i=1}^{5} a_i V^i dV = a_0 + a_2 M[V^2] + a_4 M[V^4], \qquad \text{(VI.59)}$$

since M [V] =0; M [$V^3$] =0 and M [$V^5$] =0 on the condition.

Consequently, for obtaining the approximate estimate of system is sufficient to determine 3 coefficients of $a_0$, $a_2$, $a_4$. From Table 1 (see Appendix 6) let us select the following values of nodes $\lambda_j$:

$$\lambda_1 = 0,23862;$$
$$\lambda_2 = 0,66121;$$
$$\lambda_3 = 0,93247.$$

In this case the value of statistical nodes (V.10)

$$\mu_{ll} = \frac{b-a}{2}\lambda_l + \frac{b+a}{2}, \qquad \text{(VI.60)}$$

where

$$b=\frac{1}{2};\ a=-\frac{1}{2},$$

i.e.

$$\mu_{11}=0,119305;$$
$$\mu_{22}=0,330000;$$
$$\mu_{33}=0,466230.$$

So that the coefficients $a_1$, $a_3$, $a_5$ would not affect solution, let us assume that they are found from the conditions

$$\left.\begin{aligned} a_1\mu_{11}+a_3\mu_{11}^3+a_5\mu_{11}^5=0;\\ a_1\mu_{22}+a_3\mu_{22}^3+a_5\mu_{22}^5=0;\\ a_1\mu_{33}+a_3\mu_{33}^3+a_5\mu_{33}^5=0.\end{aligned}\right\} \quad \text{(VI.61)}$$

Then it is easy to determine coefficients $a_0$, $a_2$, $a_4$, solving the system of the linear algebraic equations

$$\left.\begin{aligned} Y(t_1,\mu_{11})=a_0+a_2\mu_{11}^2+a_4\mu_{11}^4;\\ Y(t_1,\mu_{22})=a_0+a_2\mu_{22}^2+a_4\mu_{22}^4;\\ Y(t_1,\mu_{33})=a_0+a_2\mu_{33}^2+a_4\mu_{33}^4.\end{aligned}\right\} \quad \text{(VI.62)}$$

i.e.

$$a_0=\frac{\Delta_0}{\Delta};\ a_2=\frac{\Delta_2}{\Delta};\ a_4=\frac{\Delta_4}{\Delta}, \quad \text{(VI.63)}$$

where

$$\Delta=\begin{vmatrix} 1 & \mu_{11}^2 & \mu_{11}^4\\ 1 & \mu_{22}^2 & \mu_{22}^4\\ 1 & \mu_{33}^2 & \mu_{33}^4\end{vmatrix};$$

$$\Delta_0=\begin{vmatrix} Y(t_1,\mu_{11}) & \mu_{11}^2 & \mu_{11}^4\\ Y(t_1,\mu_{22}) & \mu_{22}^2 & \mu_{22}^4\\ Y(t_1,\mu_{33}) & \mu_{33}^2 & \mu_{33}^4\end{vmatrix};$$

$$\Delta_2=\begin{vmatrix} 1 & Y(t_1,\mu_{11}) & \mu_{11}^4\\ 1 & Y(t_1,\mu_{22}) & \mu_{22}^4\\ 1 & Y(t_1,\mu_{33}) & \mu_{33}^4\end{vmatrix};$$

$$\Delta_4=\begin{vmatrix} 1 & \mu_{11}^2 & Y(t_1,\mu_{11})\\ 1 & \mu_{22}^2 & Y(t_1,\mu_{22})\\ 1 & \mu_{33}^2 & Y(t_1,\mu_{33})\end{vmatrix}.$$

Page 170.

Further remains to substitute the obtained values of coefficients of $a_0$, $a_2$, $a_4$ into formula (VI.59) for determining approximate value of the mathematical expectation M [Y] at different moments of time. Table VI.1 gives for the comparison exact and approximate values of calculations. The results of computations testify about the high accuracy of method.

3. Method of the formation of the exciting interactions through the equations of relation.

Let on system act random variables $V_1, V_2, \ldots, V_m$, which determine the m-dimensional region S of statistical nodes, for example, A, B, C (Fig. VI.2). As it was noted above, one of the ways of the determination of the statistical evaluations of system [16, 34, 107] is the path of the formation of random disturbances, the assignment of statistical nodes (VI.5), which determine the behavior of system.

Method considered/examined below makes it possible in certain cases to carry out formation of random variables with the help of one random variable, which makes it possible to substantially facilitate

the solution of the problem of research, complex nonlinear systems with a large number of random interactions due to the decrease of the space of integrations.

Let us assume that for the concrete definition of stated problem system A has m of random interactions (Fig. VI.3).

Table VI.1.

| t сек (1) | M [Y] | $\tilde{M}$ [Y] | (2) Погрешность в % |
|---|---|---|---|
| 0,0 | 0,0000 | 0,0000 | 0 |
| 0,1 | 0,0999 | 0,1002 | ~0,3 |
| 0,2 | 0,1999 | 0,2004 | ~0,25 |
| 0,5 | 0,4982 | 0,4981 | ~0,02 |
| 1,0 | 0,9862 | 0,9867 | ~0,05 |

Key: (1). s. (2). Error.

Page 171.

Let us introduce new random variables $X_1, X_2, \ldots, X_m,$ connected with initial random variables $V_1, V_2, \ldots, V_m$ with some functions of the known form

$$\left.\begin{aligned} X_1 &= f_1(V_1, V_2, \ldots, V_m); \\ X_2 &= f_2(V_1, V_2, \ldots, V_m); \\ X_m &= f_m(V_1, V_2, \ldots, V_m) \end{aligned}\right\} \quad \text{(VI.64)}$$

also, between themselves by the following relations:

$$\left.\begin{aligned} X_1 &= \varphi_1(X_l); \\ X_2 &= \varphi_2(X_l); \\ &\cdots\cdots\cdots \\ X_{l-1} &= \varphi_{l-1}(X_l); \\ X_{l+1} &= \varphi_{l+1}(X_l); \\ &\cdots\cdots\cdots \\ X_m &= \varphi_m(X_l) \end{aligned}\right\} \quad \text{(VI.65)}$$

or

$$\begin{aligned} X_1 &= \varphi_1(X_2); \\ X_2 &= \varphi_2(X_3); \\ &\cdots\cdots\cdots \\ X_{m-1} &= \varphi_{m-1}(X_m), \end{aligned}$$

or by others, that determine interdependency $X_1, X_2, \ldots, X_m$.

In equations (VI.64) and (VI.65) are unknown some coefficients and statistical characteristics of new random variables $X_1, X_2, \ldots, X_m$. For their determination it is necessary to compose the system of equations, which relates the statistical characteristics of new and initial random variables, that usually does not cause special difficulties.

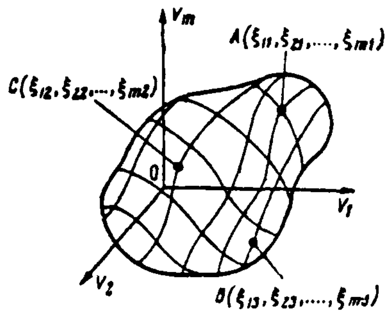



Fig. VI.2.

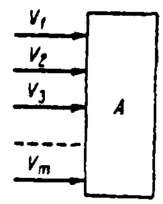



Fig. VI.3.

Fig. VI.2. Region of the assignment of statistical nodes.

Fig. VI.3. Diagram of the interaction of input disturbances/perturbations.

Page 172.

Thus, for instance, it is possible to compose the equations

$$\left.\begin{aligned} M[X_iX_j] &= M[\varphi_i\varphi_j], \ i, j = 1, 2, \ldots, m-1; \\ M[X_iX_j] &= M[f_if_j], \ i, j = 1, 2, \ldots, m; \\ M[X_mX_j] &= M[X_m\varphi_j], \ j = 1, 2, \ldots, m-1. \end{aligned}\right\} \quad \text{(VI.66)}$$

After determining the coefficients of equations (VI.64) and (VI.65) and the statistical characteristics of random variables $X_1, X_2, \ldots, X_m$, and also converting equations (VI.64) (VI.65) to the form

$$\left.\begin{aligned} V_1 &= \psi_1(X_i); \\ V_2 &= \psi_2(X_i); \\ &\cdots\cdots \\ V_m &= \psi_m(X_i), \end{aligned}\right\} \quad \text{(VI.67)}$$

where $X_i$ — random variable $X_1$, either $X_2$, ..., or $X_m$, it is possible, being given values of random variable $X_i$ through the equations of relation (VI.67), to form/shape general/common/total signal to system A on all m inputs (Fig. VI.4).

Thus, after obtaining one-dimensional random variable $X_i$, which influences the system along m channels, it is possible to substantially simplify research of system A. Moreover the interaction of random variable $X_i$ because of the equations of relation is equivalent to interaction m of random variables.

Formation of the exciting interaction for the correlated random variables. Let us assume that on the control system acts m of random variables $V_1, V_2, \ldots, V_m$ with the preset correlation matrix/die

$$\| R_{ij} \| = \left\| \begin{matrix} R_{11} & R_{12} & R_{13} & \ldots & R_{1m} \\ & R_{22} & R_{23} & \ldots & R_{2m} \\ & & R_{33} & \ldots & R_{3m} \\ & & & \ldots\ldots\ldots & \\ & & & \ldots & \\ & & & \ldots & R_{mm} \end{matrix} \right\|$$

and by known mathematical expectations $m_{V_1}, m_{V_2}, \ldots, m_{V_m}$ (information about the multipole moments they are absent).

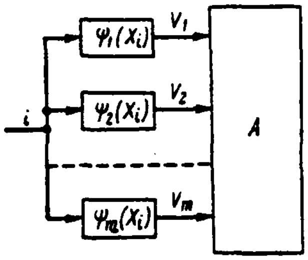



Fig. VI.4. Diagram of shaping of general/common/total input signal for system A.

Page 173.

Let us connect new random variables $X_1, X_2, \ldots, X_m$ with the initial ones and between themselves with the following system of equations:

$$\left.\begin{aligned} X_1 &= \mathring{V}_1; \\ X_2 &= a_{21}\mathring{V}_1 + \mathring{V}_2; \\ X_3 &= a_{31}\mathring{V}_1 + a_{32}\mathring{V}_2 + \mathring{V}_3; \\ &\ldots\ldots\ldots\ldots \\ X_m &= a_{m1}\mathring{V}_1 + a_{m2}\mathring{V}_2 + \ldots + a_{m,m-1}\mathring{V}_{m-1} + \mathring{V}_m; \end{aligned}\right\} \quad \text{(VI.68)}$$

$$\left.\begin{aligned} X_1 &= b_1 X_2^{g_1}; \\ X_2 &= b_2 X_3^{g_2}; \\ &\ldots\ldots \\ X_{m-1} &= b_{m-1} X_m^{g_{m-1}}, \end{aligned}\right\} \quad \text{(VI.69)}$$

where $\mathring{V}_1, \mathring{V}_2, \ldots, \mathring{V}_m$ — centered values of random variables;

$g_1, g_2, \ldots, g_{m-1}$ — odd degrees;

$X_1, X_2, \ldots, X_m$ — random variables, which have the symmetrical

distribution density.

It is not difficult to see that in system (VI.68) there is $\frac{m(m-1)}{2}$ unknown coefficients $a_{ij}$; $m-1$ of unknown coefficients $b_j$ and m of the unknown values of dispersions $D_{x_1}, D_{x_2}, \ldots, D_{x_m}$, moreover $m_{x_1}=m_{x_2}=\ldots=m_{x_m}=0$ on condition (VI.68), i.e., in all we have

$$S=\frac{m(m-1)}{2}+2m-1 \qquad \text{(VI.70)}$$

unknowns.

Consequently, after using system (VI.68), necessary to compose S of equations. Such equations will be the following:

$$\left.\begin{array}{c} D_{x_1}=\mathrm{M}\left[\overset{0}{V}{}_1^2\right]; \\ D_{x_2}=\mathrm{M}\left[(a_{21}\overset{0}{V}_1+\overset{0}{V}_2)^2\right]; \\ \cdots\cdots\cdots\cdots \\ D_{x_m}=\mathrm{M}(a_{m1}\overset{0}{V}_1+\ldots+\overset{0}{V}_m)^2 \end{array}\right\} \qquad \text{(VI.71)}$$

and

$$R_{x_ix_j}=\mathrm{M}\left[(a_{i1}\overset{0}{V}_1+\ldots+\overset{0}{V}_m)(a_{j1}\overset{0}{V}_1+\ldots+\overset{0}{V}_j)\right], \qquad \text{(VI.72)}$$
$$i\neq j,\ i,\ j=1,\ 2,\ \ldots,\ m$$

and from the system of equations (VI.69) we will have

$$\left.\begin{array}{c} D_{x_1}=\mathrm{M}\left[(b_1X_2^{g_1})^2\right]; \\ D_{x_2}=\mathrm{M}\left[(b_2X_3^{g_2})^2\right]; \\ \cdots\cdots\cdots\cdots \\ D_{x_{m-1}}=\mathrm{M}\left[\left(b_{m-1}X_m^{g_{m-1}}\right)^2\right] \end{array}\right\} \qquad \text{(VI.73)}$$

and

$$R_{x_ix_j}=\mathrm{M}\left[b_iX_{i+1}^{g_i}b_jX_{j+1}^{g_j}\right];\quad R_{x_ix_m}=\mathrm{M}\left[b_iX_{i+1}^{g_i}X_m\right], \qquad \text{(VI.74)}$$
$$i\neq j,\ i,\ j=1,\ 2,\ \ldots,\ m-1.$$

Page 174.

Equations (VI.72) and (VI.74) must be matched, i.e., must be made the equalities

$$\left.\begin{array}{c} M\left[(a_{i1}\overset{\circ}{V}_1+\ldots+\overset{\circ}{V}_i)(a_{j1}\overset{\circ}{V}_1+\ldots+\overset{\circ}{V}_j)\right]= \\ =M\left[b_iX^{g}_{i+1}b_jX^{g}_{j+1}\right]; \\ i\neq j,\ i,\ j=1,\ 2,\ \ldots,\ m-1; \\ M\left[(a_{i1}\overset{\circ}{V}_1+\ldots+\overset{\circ}{V}_i)(a_{m1}\overset{\circ}{V}_1+\ldots+\overset{\circ}{V}_m)\right]= \\ =M\left[b_iX^{g}_{i+1}X_m\right], \\ i=1,\ 2,\ \ldots,\ m-1. \end{array}\right\} \qquad (VI.75)$$

And (VI.73) it is evident from equalities (VI.71) that a number of equations is equal m+m-1=2m-1. Equalities (VI.75), obtained from equations (VI.72) and (VI.74), must form the missing part of the equations. Let us ascertain that this thus.

Let us compose the matrix/die

$$\|R_{x_ix_j}\|=\left\|\begin{array}{ccccc} R_{x_1x_1} & R_{x_1x_2} & R_{x_1x_3} & \ldots & R_{x_1x_m} \\ & R_{x_2x_2} & R_{x_2x_3} & \ldots & R_{x_2x_m} \\ & & R_{x_3x_3} & \ldots & R_{x_3x_m} \\ & & \ldots & \ldots & \ldots \\ & & & \ldots & \ldots \\ & & & & R_{x_mx_m} \end{array}\right\|.$$

Since $R_{ij}=R_{ji}$, then, as can be seen from matrix/die, it is possible to obtain $\frac{m^2-m}{2}$ equations (VI.75).

Thus, the total number of equations (VI.71) (VI.73) and (VI.75) comprises $2m-1+\frac{m^2-m}{2}$ and it is equal to a number of unknown parameters of system of equations (VI.68) (VI.69).

Page 175.

For the illustration of obtaining equations (VI.71) (VI.73) and (VI.75) let us consider the case, when on the control system act 3 the random variables $V_1$, $V_2$, $V_3$, and the equations of relation take the form

$$\begin{aligned}
X_1 &= \overset{0}{V}_1;\\
X_2 &= a_{21}\overset{0}{V}_1 + \overset{0}{V}_2;\\
X_3 &= a_{31}\overset{0}{V}_1 + a_{32}\overset{0}{V}_2 + \overset{0}{V}_3;\\
X_1 &= b_1 X_1^3;\\
X_2 &= b_2 X_3^5.
\end{aligned}$$

Let the random variables $V_1$, $V_2$, $V_3$ have the correlation matrix/die

$$\| R_{ij} \| = \begin{Vmatrix} R_{11} & R_{12} & R_{13} \\ & R_{22} & R_{23} \\ & & R_{33} \end{Vmatrix}$$

Then the unknown equations can be recorded in the form

$$D_{x_1} = R_{11};$$
$$D_{x_2} = a_{21}^2 R_{11} + 2a_{21} R_{12} + R_{22};$$
$$D_{x_3} = a_{31}^2 R_{11} + a_{32}^2 R_{22} + R_{33} + 2a_{31} a_{32} R_{12} + 2a_{31} R_{13} + 2a_{32} R_{23};$$
$$D_{x_1} = b_1^2 \mathrm{M}\left[X_2^6\right];$$
$$D_{x_3} = b_2^2 \mathrm{M}\left[X_3^{10}\right];$$
$$a_{21} R_{11} + R_{12} = b_1 \mathrm{M}\left[X_2^4\right];$$
$$a_{31} R_{11} + a_{32} R_{12} + R_{13} = b_1 b_2^3 \mathrm{M}\left[X_3^{16}\right];$$
$$a_{21} a_{31} R_{11} + (a_{21} a_{32} + a_{31}) R_{12} + a_{21} R_{13} + R_{23} + a_{32} R_{22} = b_2 \mathrm{M}\left[X_3^6\right].$$

The obtained equations are nonlinear; therefore their solution causes definite difficulties. Will consider below some paths, which facilitate the determination of unknown coefficients $a_{ij}$ and, therefore, dispersions $D_{x_i}, i = 1, 2, \ldots, m$.

Page 176.

Example. On system act two random variables $V_1$ and $V_2$, characterized by correlation matrix/die

$$\| R_{ij} \| = \begin{Vmatrix} R_{11} R_{12} \\ R_{21} R_{22} \end{Vmatrix}$$

and mathematical ones by expectations $m_{V_1}, m_{V_2}$. Let us compose two equations of relation:

$$\left.\begin{aligned} X_1 &= V - m_{V_1} = \overset{0}{V}_1; \\ X_2 &= a_{21}(V_1 - m_{V_1}) + V_2 - m_{V_2} = a_{21}\overset{0}{V}_1 + \overset{0}{V}_2, \end{aligned}\right\} \qquad \text{(VI.76)}$$

where $X_1$, $X_2$ - new random variables;

$a_{21}$ — unknown coefficient.

Let us take also that the new random variables are connected with the relationship/ratio

$$X_1 = b_1 X_2^3, \qquad \text{(VI.77)}$$

where $X_2$ - random variable with the normal distribution law.

Let us find the values of coefficients of $a_{21}$, $b_1$ and dispersions $D_{X_1}$ for the transition/junction to investigation of system with one random variable.

It follows from equations (VI.76) that the mathematical expectations of the random variables $X_1$ and $X_2$ are equal to zero, i.e., $m_{X_1} = m_{X_2} = 0$. This condition satisfies also equality (VI.77), since the random variable $X_2$ has the normal law of distribution

$$M[X_1] = M\left[b_1 X_2^3\right] = b_1 M\left[X_2^3\right] = 0.$$

Then, using equations (VI.71) (VI.73) (VI.75), it is possible to obtain the following system:

$$\left.\begin{aligned}
D_{X_1} &= M\left[\overset{0}{V}{}_1^2\right] = R_{11};\\
D_{X_2} &= M\left[\left(a_{21}\overset{0}{V}_1 + \overset{0}{V}_2\right)^2\right] = a_{21}^2 R_{11} + 2a_{21}R_{12} + R_{22};\\
R_{X_1X_2} &= M\left[\overset{0}{V}_1\left(a_{21}\overset{0}{V}_1 + \overset{0}{V}_2\right)\right] = a_{21}R_{11} + R_{12};\\
R_{X_1X_2} &= M\left[b_1 X_2^3 X_2\right] = 3b_1 D_{X_2}^2;\\
D_{X_1} &= M\left[b_1^2 X_2^6\right] = 15 b_1^2 D_{X_2}^3.
\end{aligned}\right\} \qquad \text{(VI.78)}$$

From equations (VI.78) let us construct four nonlinear algebraic equations

$$\left.\begin{aligned} D_{X_1} &= R_{11}; \\ D_{X_2} &= a_{21}^2 R_{11} + 2a_{21}R_{12} + R_{22}; \\ a_{21}R_{11} + R_{12} &= 3b_1 D_{X_2}^2; \\ D_{X_1} &= 15b_1^2 D_{X_2}^3 \end{aligned}\right\} \qquad (VI.79)$$

with four unknowns $D_{X_1}, D_{X_2}, a_{21}, b_1$. Solution of system gives the following expressions for determining the unknowns:

$$D_{X_1} = R_{11};$$

$$D_{X_2} = \left[\frac{-2R_{12} \pm \sqrt{6(R_{11}R_{22} - R_{12}^2)}}{2R_{11}}\right]^2 R_{11} + 2\left[\frac{-2R_{12} \pm \sqrt{6(R_{11}R_{22} - R_{12}^2)}}{2R_{11}}\right] R_{12} + R_{22};$$

$$a_{21} = \frac{-2R_{12} \pm \sqrt{6(R_{11}R_{22} - R_{12}^2)}}{2R_{11}};$$

$$b_1 = \frac{\pm\sqrt{6(R_{11}R_{22} - R_{12}^2)}}{3\left[\frac{-2R_{12} \pm \sqrt{6(R_{11}R_{22} - R_{12}^2)}}{2R_{11}}\right]^2 R_{11} + 2\left[\frac{-2R_{12} \pm \sqrt{6(R_{11}R_{22} - R_{12}^2)}}{2R_{11}}\right] R_{12} + R_{22}}.$$

Page 177.

It is necessary to have in mind that the sign before expression $\sqrt{6(R_{11}R_{22} - R_{12}^2)}$ is determined by the sign of correlation $R_{12}$.

Coefficients $a_{21}$ and $b_1$ have real values, that as the considered/examined radicand of more than zero, i.e.

$$R_{11}R_{22} - R_{12}^2 > 0. \qquad (VI.80)$$

Actually/really, let us represent expression (VI.80) in the form

$$\sigma^2_{V_1}\sigma^2_{V_2} > r^2_{V_1V_2}\sigma^2_{V_1}\sigma^2_{V_2}$$

or

$$1 > r^2_{V_1V_2}.$$

This inequality is fulfilled, that as

$$|r_{V_1V_2}| < 1.$$

The case, when $|r_{V_1V_2}| = 1$ does not have in this task of sense, since the random variables $V_1$ and $V_2$ would be connected with the relationship/ratio

$$\overset{0}{V}_2 = A\overset{0}{V}_1 + B,$$

where A and B - coefficients, and to convert to the random variables $X_1$, $X_2$ would not follow.

It is concealed by shape, coefficients $a_{21}$, $b_1$ equations (VI.76) (VI.77) and the statistical characteristics of the random variables $X_1$ and $X_2$ are found that it makes it possible to determine the structure of the converter of random variables.

Page 178.

For this let us rewrite expressions (VI.76) taking into account equality (VI.77) in the following form:

$$V_1 = X_1 + m_{V_1};$$
$$V_2 = \sqrt[3]{\frac{1}{b_1} X_1} - a_{21} X_1 + m_{V_2}$$

or

$$\left.\begin{aligned} V_1 &= X_1 + m_{V_1}; \\ V_2 &= c_1 \sqrt[3]{X_1} - c_2 X_1 + m_{V_2}, \end{aligned}\right\} \qquad \text{(VI.81)}$$

where

$$c_1 = \sqrt[3]{\frac{1}{b_1}}; \quad c_2 = a_{21},$$

and also

$$\left.\begin{aligned} V_1 &= b_1 X_2^3 + m_{V_1}; \\ V_2 &= X_2 - a_{21} b_1 X_2^3 + m_{V_2} \end{aligned}\right\} \qquad \text{(VI.82)}$$

or

$$\left.\begin{aligned} V_1 &= b_1 X_2^3 + m_{V_1} \\ V_2 &= X_2 - c_2 b_1 X_2^3 + m_{V_2}. \end{aligned}\right\} \qquad \text{(VI.83)}$$

The structural schemes, which realize relationships/ratios (VI.81) and (VI.83), are depicted in Fig. VI.5 and VI.6 respectively.

In a number of cases should be in the equations of relation (VI.76) input/embedded the relationship/ratio

$$X_2 = b_1 X_1^3,$$

where $X_1$ - random variable with the normal distribution law.

In this case equations (VI.79) will take the form

$$D_{X_1} = R_{11};$$
$$D_{X_2} = a_{21}^2 R_{11} + 2a_{21} R_{12} + R_{22};$$
$$a_{21} R_{11} + R_{12} = 3b_1 D_{X_1}^2;$$
$$D_{X_2} = 15 b_1^2 D_{X_1}^2.$$

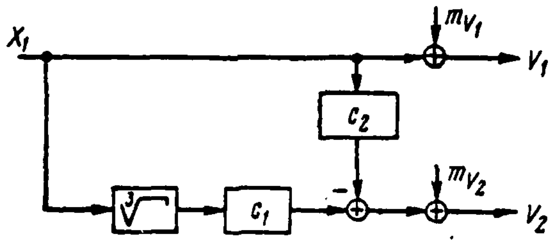



Fig. VI.5. Structural scheme, which realizes equality (VI.81).

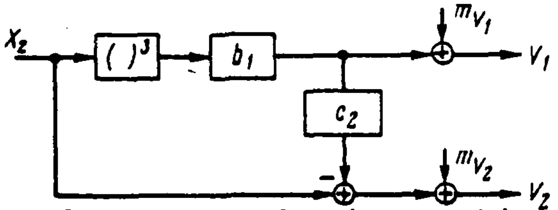



Fig. VI.6. Structural scheme, which realizes equality (VI.83).

Page 179.

Formation of the exciting interaction for the uncorrelated random variables. The formation of the random effect on the system, subjected to the effect of random variables $V_1, V_2, \ldots, V_m$ with the correlation matrix/die

$$\| R_{ij} \| = \left\| \begin{matrix} R_{11} & 0 & 0 & \ldots & 0 \\ & R_{22} & 0 & \ldots & 0 \\ & & R_{33} & \ldots & 0 \\ & & \ldots & \ldots & \ldots \\ & & & \ldots & R_{mm} \end{matrix} \right\| \qquad \text{(VI.84)}$$

employing the procedure, presented above, significantly is facilitated. This is determined mainly by simplification in equations (VI.72), (VI.74) and respectively by the simpler procedure of their solution.

It is necessary to keep in mind that if random variables $V'_1, V'_2, \ldots, V'_m$ are correlated, transition/junction to mutually uncorrelated random variables $V_1, V_2, \ldots, V_m$ does not cause difficulties. For this it is possible to use the proposed by V. S. Pugachev [77] methodology of the composition of the linear functions

$$\left.\begin{aligned} V'_1 - m_{V_1} &= V_1; \\ V'_2 - m_{V_2} &= c_{21}V_1 + V_2; \\ &\ldots\ldots\ldots\ldots \\ V'_m - m_{V_m} &= c_{m1}V_1 + c_{m2}V_2 + \ldots + c_{m,\,m-1}V_{m-1} + V_m, \end{aligned}\right\} \quad \text{(VI.85)}$$

where coefficients $c_{ij}$ are equal to:

$$\left.\begin{aligned} c_{\nu 1} &= \frac{R_{\nu 1}}{D_{V_1}}; \\ c_{\nu\mu} &= \frac{1}{D_{V_\mu}}\left(R_{\nu\mu} - \sum_{\lambda=1}^{\mu-1} c_{\nu\lambda}c_{\mu\lambda}D_{V_\lambda}\right), \quad \mu = 2, 3, \ldots, \nu - 1, \end{aligned}\right\} \quad \text{(VI.86)}$$

but the dispersions of random variables $V_1, V_2, \ldots, V_m$ are determined from the formulas

$$D_{V_\nu} = R_{\nu\nu} - \sum_{\lambda=1}^{\nu-1} |c_{\nu\lambda}|^2 D_{V_\lambda}, \quad \nu = 1, 2, \ldots, m. \qquad \text{(VI.87)}$$

The structural scheme of the formation of the random interaction $X_1$ and $X_2$ during the association of two random variables $V_1$ and $V_2$ remains previous (see Fig. VI.5 and VI.6).

Page 180.

However, the values of coefficients of $c_1$ and $c_2$ will become others, that as $R_{12}=0$:

$$\left.\begin{aligned} a_{21} &= \pm\sqrt{\frac{3}{2}\cdot\frac{R_{22}}{R_{11}}};\\ b_1 &= \pm\frac{75}{4}\cdot\frac{R_{11}}{R_{22}}\sqrt{\frac{3}{2}\cdot\frac{R_{22}}{R_{11}}}.\end{aligned}\right\} \qquad \text{(VI.88)}$$

Let us pause in somewhat more detail at the methodology of the formation of the one-coordinate random interaction $X_1$ under the effect on system m of random variables $V_1, V_2, \ldots, V_m$ with the given values of dispersions $Dv_1, Dv_2, \ldots, Dv_m$.

Let us assume that the equations of relation take the form

$$\left.\begin{aligned} X_1 &= V_1;\\ X_i &= \sum_{j=1}^{i-1} a_{ij}V_j + V_i;\\ X_i &= b_i X_1^{2i-1}, \quad i=2, 3, \ldots, m,\end{aligned}\right\} \qquad \text{(VI.89)}$$

where $X_1$ - random variable, which has the normal distribution law.

For determining of unknown coefficients $a_{ij}, b_i$ and values of covariances $Rx_ix_i$ $(i = 1, 2, \ldots, m)$ let us compose system of equations

$$\left.\begin{aligned} Rx_ix_i &= \sum_{j=1}^{i-1} a_{ij}^2 Dv_j + Dv_i, \quad i=1, 2, \ldots, m;\\ Rx_ix_i &= b_i^2\mu_{(2m-1)2}, \quad i=2, 3, \ldots, m,\\ a_{i1}D_{V_1} &= b_i\mu_{(2i)}, \quad i=2, 3, \ldots, m;\\ \sum_{j=1}^{\nu-1} a_{\nu j}a_{ij}Dv_j &+ a_{i\nu}Dv_\nu = b_\nu b_i\mu_{2(\gamma+i-1)};\\ \nu = 2, 3, \ldots, m-1;\ &\gamma=1, 2, \ldots m-2;\ i=3, 4, \ldots, m,\end{aligned}\right\} \qquad \text{(VI.90)}$$

where a number of latter/last equations is determined from formula $\gamma=i-2$, $\mu_s$ — the moment/torque of the s order of random variable. The solution of system (VI.90) can be substantially simplified, if we take into account one special feature/peculiarity.

Let us compose subsystem for the association of the random variabl s $V_1$ and $V_2$:

$$\left.\begin{array}{c} R_{x_1x_1}=D_{V_1};\quad R_{x_2x_2}=b_2^2\mu_6;\; a_{21}D_{V_1}=b_2\mu_4; \\ R_{x_2x_2}=a_{21}^2D_{V_1}+D_{V_2}. \end{array}\right\} \qquad (VI.91)$$

Page 181.

After determining from subsystem (VI.91) of value $a_{21}$, $b_2$, $R_{x_1x_1}$, $R_{x_2x_2}$, let us record the new subsystem

$$\left.\begin{array}{c} R_{x_3x_3}=a_{31}^2D_{V_1}+a_{32}^2D_{V_2}+D_{V_3}; \\ R_{x_3x_3}=b_3^2\mu_{10}; \\ a_{31}D_{V_1}=b_3\mu_6; \\ a_{21}a_{31}D_{V_1}+a_{32}D_{V_2}=b_2b_3\mu_8 \end{array}\right\} \qquad (VI.92)$$

for the association of the random variables $V_1$, $V_2$, $V_3$. It is obvious that the solution of subsystem (VI.92) after the solution of subsystem (VI.91) significantly is simplified.

For the connection/attachment to the united three random variables $V_1$, $V_2$, $V_3$ of the subsequent random variables it is necessary to compose the subsystems

$$\left.\begin{aligned} R_{x_i x_i} &= \sum_{l=1}^{i-1} a_{il}^2 D_{V_l} + D_{V_i}; \\ R_{x_i x_i} &= b_i^2 \mu_{(2m-1)2}; \\ c_{i1} D_{V_1} &= b_i \mu_{(2i)}; \\ \sum_{l=1}^{\nu-1} a_{\nu l} a_{il} D_{V_l} + a_{i\nu} D_{V_\nu} &= b_\nu b_i \mu_{2(\gamma+i-1)}, \end{aligned}\right| \qquad (VI.93)$$

by giving to indices i, $\nu$, $\gamma$ the consecutively/serially following values:

$$\begin{gathered} i=4,\ \gamma=i-2=2,\ \nu=2,\ 3; \\ i=5,\ \gamma=i-2=3,\ \nu=2,\ 3,\ 4; \\ \cdots\cdots\cdots\cdots\cdots \\ i=m,\ \ \gamma=i-2=m-2,\ \ \nu=2,\ 3,\ \ldots,\ m-1. \end{gathered}$$

Thus, for i=5, $\gamma$=3, $\nu$=2, 3, 4 subsystems (VI.93) will take the form

$$\begin{gathered} R_{x_5 x_5} = a_{51}^2 D_{V_1} + a_{52}^2 D_{V_2} + a_{53}^2 D_{V_3} + c_{54}^2 D_{V_4} + D_{V_5}; \\ R_{x_5 x_5} = b_5^2 \mu_8; \\ a_{51} D_{V_1} = b_5 \mu_{10}; \\ c_{21} a_{51} D_{V_1} + c_{52} D_{V_2} = b_2 b_5 \mu_{14}; \\ a_{31} a_{51} D_{V_1} + a_{32} a_{52} D_{V_2} + a_{53} D_{V_3} = b_3 b_5 \mu_{14}; \\ a_{41} a_{51} D_{V_1} + a_{42} a_{52} D_{V_2} + a_{43} a_{53} D_{V_3} + c_{54} D_{V_4} = b_4 b_5 \mu_{14}. \end{gathered}$$

In subsystem (VI.93) the first two equations are square, and rest, in connection with the presence of the solutions of the previous subsystems, linear.

Page 182.

After solving system (VI.90), it is possible to easily find

expressions for random variables $V_1, V_2, \ldots, V_m$ through $X_1$:

$$V_l = \frac{\begin{vmatrix} 1 & 0 & \cdots & 0 & X_1 & 0 & \cdots & 0 \\ a_{21} & 1 & \cdots & 0 & b_2X_1^3 & \vdots & \cdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{m-1} & a_{m-2} & \cdots & a_{m,l-1} & b_mX_1^{2m-1} & a_{m,l+1} & \cdots & a_{mm} \end{vmatrix}}{\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_2 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & & & & \\ a_{m-1} & a_{m-2} & \cdots & a_{m,l-1} & a_{ml} & a_{m,l+1} & \cdots & a_{mm} \end{vmatrix}} \quad \text{(VI.94)}$$

Example. Find the value of the coefficients of the equations of relation

$$X_1 = V_1;$$
$$X_2 = a_{21}V_1 + V_2;$$
$$X_3 = a_{31}V_1 + a_{32}V_2 + V_3;$$
$$X_2 = b_2X_1^3;$$
$$X_3 = b_3X_1^5$$

for the expression of the uncorrelated random variables $V_1$, $V_2$, $V_3$ through $X_1$, if $D_{V_1} = 6$; $D_{V_2} = 1$; $D_{V_3} = 3$.

After composing systems (VI.91) and (VI.92) and after solving them, let us find expressions for determining the unknowns

$$R_{X_1X_1} = D_{V_1};$$
$$b_2 = \frac{1}{D_{V_1}} \sqrt{\frac{D_{V_2}}{6D_{V_1}}};$$
$$a_{21} = 3 \sqrt{\frac{D_{V_2}}{6D_{V_1}}};$$
$$R_{X_2X_2} = \frac{5}{2} D_{V_2};$$
$$a_{31} = \sqrt{\frac{15}{8} \cdot \frac{D_{V_3}}{D_{V_1}}};$$
$$b_3 = \frac{1}{D_{V_1}^2} \sqrt{\frac{D_{V_3}}{120D_{V_1}}};$$
$$R_{X_3X_3} = \frac{63}{8} D_{V_3};$$
$$a_{32} = \sqrt{5 \frac{D_{V_3}}{D_{V_2}}}.$$

Page 183.

After determining the values of the parameters given above, let us record reference system in the form

$$X_1 = V_1;$$
$$X_2 = \frac{1}{2} V_1 + V_2;$$
$$X_3 = \frac{1}{4} \sqrt{15} V_1 + \sqrt{15} V_2 + V_3;$$
$$X_2 = \frac{1}{36} X_1^3; \; X_3 = \frac{1}{2160} \sqrt{15} X_1^5.$$

Is further easy to obtain expressions (VI.92) for determining of $V_1$, $V_2$, $V_3$ through $X_1$:

$$V_1 = X_1;$$
$$V_2 = \frac{1}{36} X_1^3 - \frac{1}{2} X_1;$$
$$V_3 = \sqrt{15}\left(\frac{1}{2160} X_1^5 - \frac{1}{36} X_1^3 + \frac{1}{4} X_1\right).$$

Correlation matrix/die of the random variables $X_1$, $X_2$, $X_3$

$$\| R_{ij} \| = \left\| \begin{array}{ccc} 6 & 3 & \frac{3}{2}\sqrt{15} \\ & \frac{5}{2} & \frac{7}{4}\sqrt{15} \\ & & \frac{189}{8} \end{array} \right\|.$$

It is easy to check that the random variables will be uncorrelated. Actually

$$R_{V_1V_2} = \mathrm{M}\left[X_1\left(X_2 - X_1\frac{1}{2}\right)\right] = R_{X_1X_2} - \frac{1}{2} R_{X_1X_1} = 0;$$
$$R_{V_1V_3} = \mathrm{M}\left[X_1\left(X_3 - \sqrt{15}X_2 + \frac{1}{4}\sqrt{15}X_1\right)\right] =$$
$$= R_{X_1X_3} - \sqrt{15}R_{X_1X_2} + \frac{1}{4}\sqrt{15}R_{X_1X_1} = 0;$$
$$R_{V_2V_3} = \mathrm{M}\left[\left(X_2 - \frac{1}{2}X_1\right)\left(X_3 - \sqrt{15}X_2 + \frac{1}{4}\sqrt{15}X_1\right)\right] =$$
$$= R_{X_2X_3} - \sqrt{15}R_{X_2X_2} + \frac{1}{4}\sqrt{15}R_{X_2X_1} - \frac{1}{2}R_{X_1X_3} + \frac{1}{2}\sqrt{15}R_{X_1X_2} -$$
$$- \frac{1}{8}\sqrt{15}R_{X_1X_1} = 0.$$

Page 184.

# CHAPTER VII.

## The approximation method of determining the density of distribution of probabilities of the output coordinates of nonlinear systems.

### 1. Formulation of the problem and essence of method.

Let the behavior of automatic control system be as before described by the system of the nonlinear differential equations of the n order:

$$\frac{dY_j}{dt} = f_j(Y_1, \ldots, Y_n, V_1, \ldots, V_m, t) \quad \text{(VII.1)}$$
$$(j = 1, \ldots, n)$$

or in the vector form

$$\frac{dY}{dt} F(Y, V, t),$$

where

$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$ - the vector of output coordinates;

$V=\begin{pmatrix}V_1\\ \vdots\\ V_m\end{pmatrix}$ - vector of the random parameters;

$F(Y, V, t)=\begin{pmatrix}f_1(Y, V, t)\\ \vdots\\ f_n(Y, V, t)\end{pmatrix}$ - vector of right sides;

$f_i(Y, V, t)$ - some (in the general case nonlinear) continuous functions of time, coordinates and random variables.

Page 185.

Let us assume known

$p(y_{1(t)}, \ldots, y_{n(t)})$ - the combined density of the distribution of the probabilities of the output coordinates of system $Y_1, \ldots, Y_n$ at certain fixed/recorded moment/torque $t = t_i$; in the following presentation will be sometimes designated $p(y_{(t)})$;

$p_V(v_1, \ldots, v_m)$ - the combined density of the distribution of the probabilities of random parameters $V_1, \ldots, V_m$.

Task lies in the fact that to find the density of the distribution of the probabilities of output coordinates in any fixed/recorded moment of time $t \in [t_i, T]$, where $[t_i, T]$ - the time interval

in question.

If we designate through $Y_{(i)}$ the value of vector $Y$ at moment/torque $t_{(i)}$, then for value $Y_{(i+1)}$ of vector $Y$ at moment/torque $t_{i+1} = t_i + \Delta t$ it is possible to record the following expression:

$$Y_{(i+1)} = Y_{(i)} + \Delta Y_{(i)}, \qquad \text{(VII.2)}$$

where $\Delta Y_{(i)} = \begin{pmatrix} \Delta Y_{1(i)} \\ \vdots \\ \Delta Y_{n(i)} \end{pmatrix}$ - vector of an increment in the output coordinates for the time Δt.

Using known formulas, it is possible to record expressions for the joint probability density of the components of vector $Y_{(i+1)}$, as for the probability density of the sum of vectors $Y_{(i)}$ and $\Delta Y_{(i)}$ in the following form:

$$p\left(y_{1(i+1)}, \ldots, y_{n(i+1)}\right) = \int \overset{\infty}{\underset{-\infty}{\overset{(m)}{\cdots}}} \int p\left(y_{1(i)}, \ldots, y_{n(i)}\right) \times$$
$$\times p_{\Delta}\left(y_{1(i+1)} - y_{1(i)}, \ldots, y_{n(i+1)} - y_{n(i)} / y_{1(i)}, \ldots, y_{n(i)}\right) \times$$
$$\times dy_{1(i)} \ldots dy_{n(i)}. \qquad \text{(VII.3)}$$

Here $p_{\Delta}\left(y_{1(i+1)} - y_{1(i)}, \ldots, y_{n(i+1)} - y_{n(i)} / y_{1(i)}, \ldots, y_{n(i)}\right)$ the conditional probability density of increment $\Delta Y_{(i)}$ with that fixed/recorded $Y_{(i)}$, where instead of $\Delta y_{\kappa(i)} (i = 1, \ldots, n)$ is substituted $y_{\kappa(i+1)} - y_{\kappa(i)}$.

Let us record expression for $\Delta Y$ in the form of Taylor series

$$\Delta Y_{(i)} = F(Y_{(i)}, V, t_i)\,\Delta t + \frac{dF(Y_{(i)}, V, t_i)}{dt}\cdot\frac{\Delta t^2}{2} + \cdots$$

With sufficiently small Δt it is possible to be bounded to the first term of this expansion, then

$$\Delta Y_{(i)} = F(Y_{(i)}, V, t_i)\,\Delta t. \qquad \text{(VII.4)}$$

Page 186.

The task of determining the conditional probability density of the vector of increments was reduced, thus, to the task of determining the probability density of the nonlinear determined function from the random parameters. The complexity of its solution is determined by the form of the function $F(Y, V, t)$ and by the law of distribution of the random parameters. In the general case the unknown conditional probability density can be determined by computing the multiple integral. In special cases it is possible to find evident expression for the conditional probability density of the vector of the increments through the probability density of the random parameters.

The idea of the method, set forth in present chapter, consists in the fact that after the separation of the interval in question into series/row of sections with the step/pitch Δt consistently is

determined from formula (VII.3) the density of the distribution of the probabilities of the coordinates of system at each point of separation, i.e. on the preset probability density at the initial moment $t_0$ probability density at moment/torque $t_1$ is determined, on the probability density at moment/torque $t_1$ is determined the probability density at moment/torque $t_2$, and so forth. In this case into formula (VII.3) instead of $p_\Delta$ its expression, obtained by some method or other, is substituted.

The greatest difficulties in computational sense during the use of this method are connected with the need of computing the multiple integrals on TsVM [ЦВМ - digital computer] and with the approximation of the functions of many variables. Therefore at present the use/application of the method in question is appropriate with the low order of the system of the differential equations, which describe the behavior of automatic control system.

Some special features/peculiarities of the computation of the multiple integrals, which appear during the use of an approximation method, and possibilities of different representation of probability density as the functions of many variables are presented in p. 5 of this chapter.

2. General case.

Let us consider the general case. $p_V(v_1, ..., v_m)$ - joint probability density of the random parameters.

Taking into account equation (VII.4), following recommendations of V. S. Pugachev [77], it is possible to obtain the following expression for the conditional probability density of the vector of the increments:

$$p_\Delta(\Delta y_{1(i)}, \ldots, \Delta y_{n(i)}/y_{1(i)}, \ldots, y_{n(i)}) = p_\Delta(\Delta y_{(i)}/y_{(i)}) = \\ = \int_{-\infty}^{\infty} \overset{(m)}{\ldots} \int p_V(v_1, \ldots, v_m)\, \delta\left[\Delta y_{1(i)} - \eta_{1(i)}\right] \ldots \delta\left[\Delta y_{n(i)} - \eta_{n(i)}\right] dv_1 \ldots dv_m. \quad \text{(VII.5)}$$

where

$$\underset{(j=1,\ldots,n)}{\eta_{j(i)}} = f_j(y_{1(i)}, \ldots, y_{n(i)}, v_1, \ldots, v_m)\, \Delta t = \\ = f_j(y_1, \ldots, y_n, v_1, \ldots, v_m, t_i)\, \Delta t, \quad \text{(VII.6)}$$

is the $\delta[\ldots]$ - $\delta$- function.

Page 187.

If with n<m equation (VII.6) are solved relatively $v_1, ..., v_m$

$$v_r \quad \psi_r(\eta_{1(i)}, \ldots, \eta_{n(i)}, v_{n+1}, \ldots, v_m), \quad \text{(VII.7)}$$

then replacement of variables in integral (VII.5) and integration on $\eta_{1(l)}, \ldots, \eta_{n(l)}$ with the use of properties of δ-functions gives the following expression for the conditional probability density of the vector of increments, in which variables $\eta_{1(l)}, \ldots, \eta_{n(l)}$ are replaced on $\Delta y_{1(l)}, \ldots, \Delta y_{n(l)}$:

$$p_{\Delta}(\Delta y_{(l)}/y_{(l)}) = \int\limits_{-\infty}^{\infty} \overset{(m-n)}{\cdots} \int p_V(\psi_1, \ldots, \psi_n, v_{n+1}, \ldots, v_m) \times \times \left| \frac{\partial(\psi_1, \ldots, \psi_n)}{\partial(\Delta y_{1(l)}, \ldots, \Delta y_{n(l)})} \right| dv_{n+1} \ldots dv_m, \tag{VII.8}$$

where

$$\frac{\partial(\psi_1, \ldots, \psi_n)}{\partial(\Delta y_{1(l)}, \ldots, \Delta y_{n(l)})} = \begin{vmatrix} \frac{\partial\psi_1}{\partial\Delta y_{1(l)}} & \cdots & \frac{\partial\psi_1}{\partial\Delta y_{n(l)}} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \frac{\partial\psi_n}{\partial\Delta y_{1(l)}} & \cdots & \frac{\partial\psi_n}{\partial\Delta y_{n(l)}} \end{vmatrix} \text{— Jacobian}$$

Substituting relationship (VII.8) in expression (VII.3), we will obtain

$$p(y_{1(l+1)}, \ldots, y_{n(l+1)}) = \int\limits_{-\infty}^{\infty} \overset{(n)}{\cdots} \int p(y_{1(l)}, \ldots, y_{n(l)}) \times \times \left\{ \int\limits_{-\infty}^{\infty} \overset{(m-n)}{\cdots} \int p_V(\psi_1, \ldots, \psi_m, v_{n+1}, \ldots, v_m) \times \times \left| \frac{\partial(\psi_1, \ldots, \psi_n)}{\partial(\Delta y_{1(l)}, \ldots, \Delta y_{n(l)})} \right| dv_{n+1} \ldots dv_m \right\} dy_{1(l)} \ldots dy_{n(l)}. \tag{VII.9}$$

Page 188.

If with n=m equations (VII.6) have the unique solution

$$u_r = \psi_r(\eta_{1(i)}, \ldots, \eta_{n(i)}), \quad (r = 1, \ldots, n) \qquad \text{(VII.10)}$$

then the replacement of variables in integral (VII.5) and integration on $\eta_{1(i)}, \ldots, \eta_{n(i)}$ with the use of properties of δ-functions they give in to the expression

$$p_\Delta(\Delta y_{(i)}/y_{(i)}) = p_V(\psi_1, \ldots, \psi_n)\left|\frac{\partial(\psi_1, \ldots, \psi_n)}{\partial(\Delta y_{1(i)}, \ldots, \Delta y_{n(i)})}\right|. \qquad \text{(VII.11)}$$

Now expression (VII.3) can be represented in the form

$$p(y_{1(i+1)}, \ldots, y_{n(i+1)}) = \int\overset{\infty}{\underset{-\infty}{\overset{(n)}{\cdots}}}\int p(y_{1(i)}, \ldots, y_{n(i)}) \times \times p_V(\psi_1, \ldots, \psi_n)\left|\frac{\partial(\psi_1, \ldots, \psi_n)}{\partial(\Delta y_{1(i)}, \ldots, \Delta y_{n(i)})}\right| dy_{1(i)} \ldots dy_{n(i)}. \qquad \text{(VII.12)}$$

If n=m, then analogously it is possible to obtain the following expression:

$$p(y_{1(i+1)}, \ldots, y_{n(i+1)}) = \int\overset{\infty}{\underset{-\infty}{\overset{(n)}{\cdots}}}\int p(y_{1(i)}, \ldots, y_{n(i)})\, p_V(\psi_1, \ldots, \psi_n) \times \times\left|\frac{\partial(\psi_1, \ldots, \psi_n)}{\partial(\Delta y_{1(i)}, \ldots, \Delta y_{n(i)})}\right| \delta(\Delta y_{(n+1)(i)} - \psi_{(n+1)(i)}) \ldots \times \times\delta(\Delta y_{m(i)} - \psi_{m(i)})\, dy_{1(i)} \ldots dy_{m(i)}. \qquad \text{(VII.13)}$$

Research of the system, described by differential first-order equation with several random parameters

$$\frac{dY}{dt} = f(Y, V_1, \ldots, V_m, t). \qquad \text{(VII.14)}$$

is a special case of using formula (VII.9).

It is not difficult to see that in this case the probability density of coordinate Y at moment/torque $t_{i+1} = t_i + \Delta t$ is determined by the formula

$$p(y_{(i+1)}) = \int_{-\infty}^{\infty} p(y_{(i)}) \left\{ \int \overset{(m-1)}{\underset{-\infty}{\overset{\infty}{\cdots}}} \int p_V(\psi_1, v_2, \ldots, v_m) \times \right.$$
$$\left. \times \left| \frac{\partial \psi_1}{\partial \Delta y_{(i)}} \right| dv_2 \ldots dv_m \right\} dy_{(i)}. \qquad \text{(VII.15)}$$

If m=1, then

$$p(y_{(i+1)}) = \int_{-\infty}^{\infty} p(y_{(i)}) p_V(\psi_1) \left| \frac{\partial \psi_1}{\partial \Delta y_{(i)}} \right| dy_{(i)}. \qquad \text{(VII.16)}$$

Thus, formulas (VII.9) and (VII.12) (VII.13) (VII.15) or (VII.16) allow on the preset probability density of vector Y at moment/torque $t_i$ and the probability density of the random parameters of system to determine the probability density of vector at moment/torque $t_{i+1} = t_i + \Delta t$. Knowing density at moment/torque $t_{i+1}$, it is possible to determine it at moment/torque $t_{i+2} = t_{i+1} + \Delta t$ and so forth.

Page 189.

Knowing the joint probability density of the components of vector $Y$ at any moment $t_\kappa$, it is possible to obtain the probability density of each component along the formula

$$p(y_{l(\kappa)}) = \int_{-\infty}^{\infty} \overset{(n-1)}{\cdots} \int p(y_{1(\kappa)}, \ldots, y_{n(\kappa)}) \prod_{\substack{i=1 \\ i \neq l}}^{n} dy_{i(\kappa)}.$$

Thus, if we use the method examined, then the task of determining the density of the distribution of the probabilities of the vector of the coordinates of system is reduced to the consecutive computation of multiple integrals for each moment/torque $t_{i+1} (i = 0, 1, \ldots, N-1), t_{i+1} \in [t_0, T], T = t_N$.

The multiplicity of integrals in the general case is determined by an order of the system of differential equations and by a number of random parameters.

Thus, the use/application of the method indicated in the general case leads to the sufficiently cumbersome calculations, which require the expenditure of long machine time TsVM. It is possible to indicate the cases, when the use/application of the method examined leads to certain reduction of computational work during the solution of stated problem, however.

3. Special cases of applying the approximation method.

Let us consider the class of tasks, in which the system of equations (VII.1) can be represented in the form

$$\frac{dY_j}{dt} = F_{j0}(Y_1, \ldots, Y_n) + \sum_{l=1}^{m} F_{jl}(Y_1, \ldots, Y_n) V_l, \quad \text{(VII.17)}$$
$$(j = 1, \ldots, n),\ n < m,$$

where $F_{jl}(Y_1, \ldots, Y_n)$ - some nonrandom functions of the coordinates of system;

$V_l$ - mutually uncorrelated central random variables, distributed according to the normal law with dispersion $D[V_l]$.

Thus, the right side of each equation of system is the linear combination of the mutually uncorrelated central random parameters, distributed according to the normal law with the coefficients, which depend on the output coordinates of system.

Page 190.

Let us show that in this case, when system can be represented by expression (VII.17), during the use of an approximation method the multiplicity of the obtained integrals is determined by the order of system and does not depend on a number of random parameters.

If we, as before, are bounded to the first term of expansion (VII.4), then j component of vector of increments can be represented

in the form

$$\Delta Y_{j(t)} = F_{j0}(Y_{1(t)}, \ldots, Y_{n(t)})\,\Delta t + \Delta t \sum_{l=1}^{m} F_{jl}(Y_{1(t)}, \ldots, Y_{n(t)})\,V_l. \tag{VII.18}$$

Since $V_1, \ldots, V_m$ are distributed according to the normal law, then increments $\Delta Y_{j(t)} (j = 1, \ldots, n)$ with those fixed/recorded $Y_{j(t)} (j = 1, \ldots, n)$, that correspond to moment/torque $t_i$, are also distributed according to the normal law.

Thus, each component of vector of increments $\Delta Y_{(t)}$ $(j = 1, \ldots, n)$ with that fixed/recorded $Y_{(t)}$ is distributed according to the normal law with the mathematical expectation

$$M[\Delta Y_{j(t)}] = F_{j0}(y_{1(t)}, \ldots, y_{n(t)})\,\Delta t$$

and the dispersion

$$D[\Delta Y_{j(t)}] = \Delta t^2 \sum_{l=1}^{n} F_{jl}^2(y_{1(t)}, \ldots, y_{n(t)})\,D[V_l].$$

It is not difficult to see that any linear combination of the components of vector of increments is linear combination relative to random parameters $V_l (l = 1, \ldots, m)$, in consequence of which at those fixed/recorded $Y_{j(t)} (j = 1, \ldots, n)$ it is distributed according to the normal law.

It is known, [4] that if any linear combination of components of vector is distributed normally, then vector itself is also

distributed normally.

Thus, the vector of increment $\Delta Y_{(i)}$ with that fixed/recorded $Y_{(i)}$ is distributed according to the n-dimensional normal law.

The conditional probability density of the vector of increments in this case can be represented in the form

$$p_{\Delta}(\Delta y_{(i)}/y_{(i)}) = \frac{1}{(\sqrt{2\pi})^n \sqrt{|R_{(i)}|}} e^{\frac{\lambda}{2}}, \qquad \text{(VII.19)}$$

where $|R_{(i)}| = \begin{vmatrix} R_{11(i)} \dots R_{1n(i)} \\ \dots\dots\dots\dots \\ R_{n1(i)} \dots R_{nn(i)} \end{vmatrix}$ —

determinant of correlation matrix at moment/torque $t_i$;

$$\lambda = \frac{\begin{vmatrix} R_{11(i)} & \dots & R_{1n(i)} & \Delta y_{1(i)} - \mathrm{M}[\Delta y_{1(i)}] \\ \dots & \dots & \dots & \dots \\ R_{m1(i)} & \dots & R_{nn(i)} & \Delta y_{n(i)} - \mathrm{M}[\Delta y_{n(i)}] \\ \Delta y_{1(i)} - \mathrm{M}[\Delta y_{1(i)}] & \dots & \Delta y_{n(i)} - \mathrm{M}[\Delta y_{n(i)}] & 0 \end{vmatrix}}{R_{(i)}}.$$

Page 191.

Since $V_1, \dots, V_m$ - mutually uncorrelated central random variables, then mutual covariances $R_{j\varkappa(i)} = R[\Delta y_{j(i)}, \Delta y_{\varkappa(i)}]$ of the components of vector of increments with those fixed/recorded $y_{j(i)} (j = 1, \dots, n)$ at moment/torque $t_i$ are determined by the following formula:

$$R_{j\kappa(i)} = \Delta t^2 \sum_{l=1}^{m} F_{jl}(y_{1(i)}, \ldots, y_{n(i)}) F_{\kappa l}(y_{1(i)}, \ldots, y_{n(i)}) D[V_l].$$

It is possible to show that the determinant of correlation matrix/die $|R_{(i)}| > 0$, if is different from zero at least one of the following definitions:

$$\begin{vmatrix} F_{1\kappa}(y_{1(i)}, \ldots, y_{n(i)}) & F_{1l}(y_{1(i)}, \ldots, y_{n(i)}) & \ldots & F_{1q}(y_{1(i)}, \ldots, y_{n(i)}) \\ F_{2\kappa}(y_{1(i)}, \ldots, y_{n(i)}) & F_{2l}(y_{1(i)}, \ldots, y_{n(i)}) & \ldots & F_{2q}(y_{1(i)}, \ldots, y_{n(i)}) \\ \ldots & \ldots & \ldots & \ldots \\ E_{n\kappa}(y_{1(i)}, \ldots, y_{n(i)}) & F_{nl}(y_{1(i)}, \ldots, y_{n(i)}) & \ldots & F_{nq}(y_{1(i)}, \ldots, y_{n(i)}) \end{vmatrix}$$
$$\begin{pmatrix} \kappa, l, \ldots, q = 1, 2, \ldots, m; \\ \kappa \neq l, \ldots, \kappa \neq q, \ldots, l \neq q \end{pmatrix}.$$

In the majority of the real cases this condition is observed.

If $R_{j\kappa} = 0$ ($j, \kappa = 1, \ldots, n, \kappa \neq j$), then

$$p_\Delta(\Delta y_{(i)}/y_{(i)}) = \prod_{j=1}^{n} p_\Delta(\Delta y_{j(i)}/y_{(i)}), \qquad \text{(VII.20)}$$

where

$$p_\Delta(\Delta y_{j(i)}/y_{(i)}) = \frac{1}{\sqrt{2\pi D[\Delta y_{j(i)}]}} e^{-\frac{(\Delta y_{j(i)} - M[\Delta y_{j(i)}])^2}{2D[\Delta y_{j(i)}]}}$$

- conditional probability density of the components of vector of increments.

Substituting expression (VII.19) into formula (VII.3), we will obtain expression for the probability density of the vector of output

coordinates at moment/torque $t_{l+1}$

$$p(y_{1(l+1)}, \ldots, y_{n(l+1)}) \int \overset{\infty}{\underset{-\infty}{\overset{(n)}{\ldots}}} \int \frac{p(y_{1(l)}, \ldots, y_{m(l)})}{(\sqrt{2\pi})^n \sqrt{|R_{(l)}|}} e^{\frac{\lambda}{2}} dy_{1(l)} \ldots dy_{n(l)},$$

where (VII.21)

$$\lambda = \frac{\begin{vmatrix} R_{11(l)} & \ldots & R_{1n(l)} & y_{1(l+1)} - y_{1(l)} - M[\Delta y_{1(l)}] \\ \ldots & \ldots & \ldots & \ldots \\ R_{n1(l)} & \ldots & R_{nn(l)} & y_{m(l+1)} - y_{m(l)} - M[\Delta y_{n(l)}] \\ y_{1(l+1)} - y_{1(l)} - M[\Delta y_{1(l)}] & \ldots & y_{n(l+1)} - y_{n(l)} - M[\Delta y_{m(l)}] & 0 \end{vmatrix}}{|R_{(l)}|}.$$

Page 192.

If $R_{j\kappa} = 0$ $(j, \kappa = 1, \ldots, n;\ \kappa \neq j)$, then expression (VII.21) will take the form

$$p(y_{1(l+1)}, \ldots, y_{n(l+1)}) = \int \overset{\infty}{\underset{-\infty}{\overset{(n)}{\ldots}}} \int p(y_{1(l)}, \ldots, y_{m(l)}) \prod_{j=1}^{n} \times$$
$$\times p_\Delta (y_{j(l+1)} - y_{j(l)}/y_{(l)})\, dy_j. \quad \text{(VII.22)}$$

Thus, formulas (VII.21) and (VII.22) confirm the validity of confirmation about the fact that if the system can be represented in the form of expression (VII.17), then during the use of the method in question in present chapter the multiplicity of integrals will be determined by the order of system and does not depend on a number of random parameters.

Let us consider the case, when the behavior of automatic control

system is described by the differential first-order equation, which can be represented in the form

$$\frac{dY}{dt}=F_0(Y)+\sum_{l=1}^{n}F_l(Y)V_l, \qquad \text{(VII.23)}$$

where $V_l(l=1, \ldots, m)$ - as before uncorrelated central random variables, distributed according to the normal law.

On the basis of expression (VII.21) the probability density of coordinate Y at the moment of time $t_{i+1}$ is determined as follows:

$$p(y_{i+1})=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{p(y_{(i)})}{\sqrt{D[\Delta y_{(i)}]}}e^{-\frac{(y_{(i+1)}-y_{(i)}-M[\Delta y_{(i)}])^2}{2D[\Delta y_{(i)}]}}dy_i, \qquad \text{(VII.24)}$$

where $M[\Delta y_{(i)}]=F_0(y_{(i)})\Delta t;$

$$D[\Delta y_{(i)}]=\Delta t^2\sum_{l=1}^{n}F_l^2(y_{(i)})D[V_l].$$

Thus, if the behavior of system is described by the differential first-order equation, which can be substituted in the form of expression (VII.23), the determination of the probability density of output coordinate at each fixed/recorded moment/torque is reduced to the computation of simple integral independent of a number of random parameters.

In practice frequently are encountered the cases, when not all

random parameters are distributed according to the normal law.

Page 193.

For example, the dynamics of automatic control system is described by the differential equations

$$\frac{dY_j}{dt} = F_{j0}(Y_1, \ldots, Y_n) + \psi_j(Y_1, \ldots, Y_n)\Lambda + \\ + \sum_{l=1}^{m} F_{jl}(Y_1, \ldots, Y_n) V_l, \qquad \text{(VII.25)}$$

where $\psi_j(Y_1, \ldots, Y_n)$ and $F_{jl}(Y_1, \ldots, Y_n)$ $(j = 1, \ldots, n;\ l = 0, 1, \ldots, m)$ - some functions of the output coordinates of system;

$V_1, \ldots, V_m$ - mutually uncorrelated central random variables, distributed according to the normal law;

$\Lambda$ - random variable, which does not depend on $V_1, \ldots, V_m$, distributed according to the known, but different from normal, law with a probability density of $p_\Lambda(\lambda)$.

In this case increment $\Delta Y_{(i)}$ in vector Y for the time Δt can be represented in the form

$$\Delta\tilde{Y}_{(i)} = F_{(i)} + \Delta_{(i)}, \qquad \text{(VII.26)}$$

where

$$F_{(i)} = \begin{pmatrix} f_{1(i)} \\ \vdots \\ f_{n(i)} \end{pmatrix};$$

$$f_{j(i)} = F_{j0}(y_{1(i)}, \ldots, y_{n(i)})\,\Delta t + \Delta t \sum_{l=1}^{m} F_{jl}(y_{1(i)}, \ldots, y_{n(i)})\,V_l;$$

$(j = 1, \ldots, n)$

$$\Lambda_{(i)} = \begin{pmatrix} \Lambda_{1(i)} \\ \vdots \\ \Lambda_{n(i)} \end{pmatrix};$$

$$\Lambda_{j(i)} = \psi_j(y_{1(i)}, \ldots, y_{n(i)})\,\Lambda\Delta t.$$

$(j = 1, \ldots, n)$

It is possible to write the following expression for the probability density of the vector of increments $\Delta Y_{(i)}$ with that fixed/recorded $Y_{(i)}$:

$$p_{\Delta}(\Delta y_{(i)}/y_{(i)}) = \int \underset{-\infty}{\overset{\infty}{\overset{(n)}{\cdots}}} \int p_F(F_{(i)}/y_{(i)})\,p_{\Lambda}(\Delta y_{(i)} - F_{(i)}/y_{(i)})\,df_{1(i)} \ldots df_{n(i)}. \qquad \text{(VII.27)}$$

Using the known formulas of the probability theory, we will obtain

$$p_{\Lambda}(\lambda_1, \ldots, \lambda_m/y_{(i)}) = p_{\lambda_1}(\lambda_1/y_{(i)}) \prod_{l=2}^{n} \delta\left[\lambda_l - \frac{\psi_l(y_{1(i)}, \ldots, y_{n(i)})}{\psi_1(y_{1(i)}, \ldots, y_{n(i)})}\lambda_1\right]. \qquad \text{(VII.28)}$$

Page 194.

Conditional probability density $\lambda_1$ can be found from the formula

$$p_{\lambda_1}(\lambda_1/y_{(i)}) = \frac{1}{|\psi_1(y_{1(i)}, \ldots, y_{n(i)})\,\Delta t|}\,p_{\Lambda}(\Lambda^{(1)}/y_{(i)}), \qquad \text{(VII.29)}$$

where

$$\Lambda^{(1)} = \frac{\lambda_1}{\psi_1(y_{1(i)}, \ldots, y_{n(i)})\,\Delta t}.$$

After substituting dependence (VII.28) and (VII.29) into formula (VII.27), replacing in this case $\lambda_j (j = 1, \ldots, n)$ on $\Delta y_{j(i)} - f_{j(i)}$ and integrating on $\lambda_2, \ldots, \lambda_n$ taking into account the properties of δ-functions, we will obtain the following expression for the conditional probability density of the vector of the increments:

$$p_\Lambda(\Delta \boldsymbol{y}_{(i)}/\boldsymbol{y}_{(i)}) = \int\limits_{-\infty}^{\infty} p_F(f_{1(i)}, f_{2(i)}, \ldots, f_{n(i)}) \frac{p_\Lambda(\Lambda^{(2)}/\boldsymbol{y}_{(i)})}{|\psi_1(y_{1(i)}, \ldots, y_{n(i)})| \Delta t} df_{1(i)}, \quad \text{(VII.30)}$$

where

$$f_{j(i)} = \Delta y_{j(i)} - \frac{\psi_j(y_{1(i)}, \ldots, y_{n(i)})}{\psi_1(y_{1(i)}, \ldots, y_{n(i)})} (\Delta y_{1(i)} - f_{1(i)});$$

$(j = 2, \ldots, m)$

$$\Lambda^{(2)} = \frac{\Delta y_{1(i)} - f_{1(i)}}{\psi_1(y_{1(i)}, \ldots, y_{n(i)}) \Delta t}.$$

Substituting expression (VII.30) into formula (VII.3), it is possible to obtain relationship/ratio for determining the probability density of vector Y at moment/torque $t_{i+1}$:

$$p(\boldsymbol{y}_{(i+1)}) = p(y_{1(i+1)}, \ldots, y_{n(i+1)}) = \int\limits_{-\infty}^{\infty} \overset{(n)}{\cdots} \int p(y_{1(i)}, \ldots, y_{n(i)}) \times$$

$$\times \left\{ \int\limits_{-\infty}^{\infty} p_F(f_{1(i)}, f_{2(i)}, \ldots, f_{n(i)}) \frac{p_\Lambda(\Lambda^{(3)}/\boldsymbol{y}_{(i)})}{|\psi_1(y_{1(i)}, \ldots, y_{n(i)})| \Delta t} df_{1(i)} \right\} \times$$

$$\times dy_{1(i)} \ldots dy_{n(i)}, \quad \text{(VII.31)}$$

where

$$f_{j(i)} = y_{j(i+1)} - y_{j(i)} - \frac{\psi_j(y_{1(i)}, \ldots, y_{n(i)})}{\psi_1(y_{1(i)}, \ldots, y_{n(i)})} (y_{1(i+1)} - y_{1(i)} - f_{1(i)});$$

$(j = 2, \ldots, m)$

$$\Lambda^{(3)} = \frac{y_{1(i+1)} - y_{1(i)} - f_{1(i)}}{\psi_1(y_{1(i)}, \ldots, y_{n(i)}) \Delta t}.$$

Generalizing the obtained result, it is possible to do the following conclusion.

Page 195.

If the right sides of the system of differential equations can be represented in the form of the linear combination of the mutually uncorrelated central random variables, whose one part is distributed normally, and another - according to the law, different from the normal, then the use/application of the method presented leads to the fact that the multiplicity of integrals is determined by an order of system and by a number of random parameters, distributed according to the law, different from the normal.

As can be seen from, use/application approximation method outlined above in the cases examined for the systems of low order with the large number of random parameters gives the possibility to shorten the space of computations.

4. Estimation of the accuracy of approximation method.

The error in determination of probability density, which is obtained during the use of an approximation method, can be decomposed into three components.

The first of them is caused by an inaccuracy in the computation of integrals and is determined by the method of computing the multiple integrals.

The second is defined by the accuracy of the representation of probability density as the functions of several arguments and depends on the method of approximation or on the step/pitch of tables and methods of interpolation, if the unknown density is represented in the form of n-dimensional tables.

Third component is error inherent in the method.

For the evaluation/estimate of systematic error let us consider first-order equation

first.

$$\frac{dY}{dt}=f(Y, V_1, \ldots, V_m).$$

Let us record expression for Y at moment/torque $t_i+\Delta t$ in the form of Taylor series

$$Y(t_i+\Delta t)=Y(t_i)+\frac{dY(t_i)}{dt}\Delta t+Q_1, \qquad (VII.32)$$

where

$$Q_1=\frac{1}{2}\cdot\frac{d^2Y(t_i+\theta\Delta t)}{dt^2},$$

$$0\leqslant\theta<1.$$

The approximation method in question in the present chapter of determining the probability density is based on what expression (VII.32) is replaced by the following approximation formula:

$$Y^*(t_i+\Delta t)=Y(t_i)+\frac{dY(t_i)}{dt}\Delta t, \qquad (VII.33)$$

where through Y* markedly approximate value of the random variable Y at moment/torque $t_i+\Delta t$.

Page 196.

It is possible to show the validity of the following confirmation.

If we select Δt by such so that the module of remainder Q would be less than certain $\epsilon>0$, then the density curve of probability,

determined according to the approximation method, will be displaced relative to true curve to the value not more than $\varepsilon$.

Actually/really, if $|Q_i| \leqslant \varepsilon$,

$$|Y(t_i+\Delta t)-Y^*(t_i+\Delta t)| \leqslant \varepsilon.$$

In the extreme case

$$Y(t_i+\Delta t)=Y^*(t_i+\Delta t)\pm\varepsilon$$

and

$$p[y(t_i+\Delta t)]=p[y^*(t_i+\Delta t)\pm\varepsilon]. \qquad \text{(VII.34)}$$

Let us designate through $\Delta p_{max}$ the module of the maximum evaluation/estimate of the error in determination of probability density:

$$\Delta p_{max}=|p[y^*(t_i+\Delta t)\pm\varepsilon)-p[y^*(t_i+\Delta t)]|.$$

Expanding $p[y^*(t_i+\Delta t)\pm\varepsilon]$ into Taylor series and being limited to terms of first-order expansion, we will obtain expression for the module of the maximum evaluation/estimate of the systematic error:

$$\Delta p_{max}=\left|\frac{\partial p[y^*(t_i+\Delta t)]}{\partial y^*(t_i+\Delta t)}\right|\varepsilon. \qquad \text{(VII.35)}$$

Let us generalize the obtained result for the case of the system of equations of the n order, which let us record in the vector form

$$\frac{dY}{dt} = F(Y, V).$$

Let us write expression for vector Y at moment/torque $t_i + \Delta t$ in the form of Taylor series

$$Y(t_i + \Delta t) = Y(t_i) + \frac{dY(t_i)}{dt}\Delta t + Q_1, \qquad \text{(VII.36)}$$

where

$$Q_1 = \frac{\Delta t^2}{2} \cdot \frac{d^2 Y(t_i + \theta\Delta t)}{dt}, \quad 0 \leqslant \theta < 1.$$

Let us select Δt by such, in order to $Q_1 < \epsilon$, where the inequality sign means that the components of vector $Q_1$ do not exceed the appropriate components of the vector

$$\varepsilon = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix} \quad \begin{matrix} \varepsilon_j > 0 \\ \\ (j = 1, \ldots, n). \end{matrix}$$

Being limited to first-order derivative, we will obtain

$$Y^*(t_i + \Delta t) = Y(t_i) + \frac{dY(t_i)}{dt}\Delta t.$$

Page 197.

Then in the extreme case it is possible to record the following expression:

$$Y(t_i + \Delta t) = Y^*(t_i + \Delta t) \pm \varepsilon.$$

Hence it follows that the probability density of vector Y at moment/torque $t_i + \Delta t$ in the extreme case can be represented in the form

$$p\,[y(t_i + \Delta t)] \quad p\,[y^*(t_i + \Delta t) \pm \varepsilon].$$

For the maximum evaluation/estimate of systematic error it is possible to obtain the expression

$$\Delta p_{max} \quad \left| \frac{\partial p\,[y^*(t_i + \Delta t)]}{\partial y^*} \right| |\varepsilon|$$

or

$$\Delta p_{max} = \sum_{j=1}^{m} \left| \frac{\partial p\,[y^*(t_i + \Delta t)]}{\partial y_j} \right| \varepsilon_j. \qquad \text{(VII.37)}$$

Thus, formulas (VII.35) (VII.37) give the maximum evaluation/estimate of the systematic error in determination of probability density from the approximation method. This evaluation/estimate is connected with the evaluation/estimate of the remainder in formulas (VII.32) and (VII.36), which determines the accuracy in the integration of the system of differential equations for Euler's method with this step/pitch of integration Δt.

Accuracy in the integration for Euler's method with the preset step/pitch Δt can be in the most general case determined via the comparison of the results of the numerical integration of system for Euler's method with the preset step/pitch Δt with the "exact"

solution (virtually with the results of the numerical integration of system by more exact method or for Euler's method, but with the smaller step/pitch).

It is obvious that the systematic error in determination of probability density from the approximation method can be reduced by decreasing the step/pitch of solution $\Delta t$.

Natural to consider that the systematic error of one step/pitch of solution is the random variable, distributed in interval $[-\Delta\rho_{max}, +\Delta\rho_{max}]$ in the general case according to the unknown distribution law. Strictly speaking, interval depends on argument $\Delta Y$.

Let us attempt (at least it is rough) to rate/estimate accumulated error after N steps/pitches.

Let us introduce into the examination the relative systematic error in determination of probability density at one step/pitch $\delta_{M(i)}=\frac{\Delta\rho}{\rho(y_i)}$ and we will consider as the its random variable, distributed in interval $[\delta_{M\min}, \delta_{M\max}]$ and which does not depend on $Y_{(i)}$.

Page 198.

Let us further consider relative error $\delta_{H(i)}$ of the determination

of probability density at one step/pitch due to an inaccuracy in the computation of multiple integrals. We will consider its random variable, distributed in interval $[\delta_{H\min}, \delta_{H\max}]$, where $\delta_{H\min}$ AND $\delta_{H\max}$ - minimum and maximum values of the error, caused by an inaccuracy in the computation of integrals.

Finally, let us designate through $\delta_{\Pi(i)}$ the relative error in determination of probability density due to an inaccuracy in the representation it as the functions of many variables and we will consider this error the random variable, distributed in interval $[\delta_{\Pi\min}, \delta_{\Pi\max}]$, where $\delta_{\Pi\min}$ and $\delta_{\Pi\max}$ - respectively minimum and maximum values of error.

Thus, a total relative error in the determination of probability density at one step/pitch can be represented by the expression

$$\delta_{\Sigma(i)} = \delta_{M(i)} + \delta_{H(i)} + \delta_{\Pi(i)}.$$

On the basis of formula (VII.3) expression for determining the probability density on $(\kappa + 1)$ - the ohm step/pitch can be represented in the following form:

$$p(y_{(\kappa+1)})(1 + \delta_{\kappa+1}) = \int_{-\infty}^{\infty} p(y_{(\kappa)})(1 + \delta_{\kappa})\, p_{\Delta}(y_{(\kappa+1)} - y_{(\kappa)}/y_{(\kappa)})\, dy_{\kappa}. \tag{VII.38}$$

where $\delta_{\kappa+1}$ and $\delta_{\kappa}$ - relative error in determination of probability density after k + 1 and k of steps/pitches respectively. Here and

throughout by integration for the vector is implied multiple integration for its components. From formula (VII.38) it is possible to obtain the following expression:

$$\delta_{\kappa+1} = \delta_{\kappa} + \delta_{\Sigma(\kappa+1)}.$$

It is not difficult to see that accumulated error after N steps/pitches

$$\delta_N = \sum_{i=1}^{N} \delta_{\Sigma(i)} + \delta_0,$$

where $\delta_0$ - relative error of probability density at the initial moment. Subsequently we will assume $\delta_0=0$.

Page 199.

If one assumes that all components of error to all N steps/pitches are independent random quantities, then it is possible to record the following expression for the mathematical expectation and variances of error after N steps/pitches:

$$\left.\begin{aligned} M[\delta_N] &= \sum_{i=1}^{N} \{M[\delta_{M(i)}] + M[\delta_{H(i)}] + M[\delta_{\Pi(i)}]\}; \\ D[\delta_N] &= \sum_{i=1}^{N} \{D[\delta_{M(i)}] + D[\delta_{H(i)}] + D[\delta_{\Pi(i)}]\}, \end{aligned}\right\} \quad (VII.39)$$

where $M[\delta_{M(i)}]$, $M[\delta_{H(i)}]$, $M[\delta_{\Pi(i)}]$ - mathematical expectations of corresponding components of error at the i step/pitch;

$D[\delta_{M(i)}]$, $D[\delta_{H(i)}]$, $D[\delta_{\Pi(i)}]$ - dispersion of corresponding components of error at the i step/pitch.

If we assume further that the components of error have identical probabilistic characteristics, i.e., at each step/pitch.

$$\begin{gathered}
M[\delta_{M(i)}] = M[\delta_M] = \text{const}; \\
i = 1, \ldots, N \\
M[\delta_{H(i)}] = M[\delta_H] = \text{const}; \\
i = 1, \ldots, N \\
M[\delta_{\Pi(i)}] = M[\delta_\Pi] = \text{const}; \\
i = 1, \ldots, N \\
D[\delta_{M(i)}] = D[\delta_M] = \text{const}; \\
i = 1, \ldots, N \\
D[\delta_{H(i)}] = D[\delta_H] = \text{const}; \\
i = 1, \ldots, N \\
D[\delta_{\Pi(i)}] = D[\delta_\Pi] = \text{const}, \\
i = 1, \ldots, N
\end{gathered}$$

then

$$\left.\begin{aligned}
M[\delta_N] &= N\{M[\delta_M] + M[\delta_H] + M[\delta_\Pi]\}; \\
D[\delta_N] &= N\{D[\delta_M] + D[\delta_H] + D[\delta_\Pi]\}.
\end{aligned}\right\} \quad (\text{VII.40})$$

A number of steps/pitches of solution is determined by the formula

$$N = \frac{T}{\Delta t}.$$

Hence it follows that the decrease of the step/pitch of solution, although it reduces systematic component of error, can lead to an increase in accumulated error, if the weight of other components is commensurated with the systematic error.

Thus, the given above reasonings make it possible to determine in the first approximation, the most appropriate step/pitch of solution. For this it is necessary to come to light/detect/expose the dependence of systematic error on the step/pitch of integration, i.e. to obtain

$$e_j = e_j(\Delta t) \qquad (j = 1, \ldots, m).$$

This is reached by integrating the reference system of differential equations for Euler's method with the different step/pitch of the integration Δt.

Page 200.

From the obtained dependence it is necessary to switch over to determination $\delta_M(\Delta t)$. In this case limiting values are taken as the respectively equal ones to

$$\delta_{M\,\max} = \left(\frac{\Delta p_{\max}}{p\,(y)}\right)_{\max};$$

$$\delta_{M\,\min} = \left(\frac{\Delta p_{\min}}{p\,(y)}\right)_{\min}.$$

During determination $\delta_{Mmax}$ and $\delta_{Mmin}$ it is necessary to be oriented to the known initial probability density of output coordinates.

It is further necessary to assign some limiting values of other two components of error and law of distribution of all components of errors within the obtained maximum limits (for example, by normal law).

According to formulas (VII.40) are obtained dependences $M[\delta_N]$ and $D[\delta_N]$ on the step/pitch of solution Δt. Investigating the obtained dependences, the most advantageous value Δt in the first approximation, is chosen.

5. Special features/peculiarities of the computation of multiple integrals and representation of multidimensional probability density.

From the previous paragraphs it is evident that during the use of an approximation method the task of determining the probability density is reduced to the consecutive computation of multiple integrals. In this case, as it will be shown below, the accuracy of the computation of multiple integrals substantially affects the total error in determination of probability density.

At present there is a series/row of works [15, 20, 52, 97], dedicated to the computation of multiple integrals. In these works the necessary formulas for calculating the multiple integrals, obtained by different methods, are contained and evaluation/estimate of the accuracy of formulas is given.

At present it is not possible to give single-valued recommendations regarding the selection of one or the other formula for any case. It is obvious, the selection of calculation formula depends on specific problem.

However, the integrals, which it is necessary to calculate during the use of the method in question, have one general/common/total special feature/peculiarity, which consists in the fact that the integrand considerably differs from zero only in the small region. The latter is determined by the intersection of two regions, in one of which is substantially excellent from zero values of probability density at the i point, and in another - the value of exponential curve.

Let us designate through $b_{j(i)}^{(-)}$ and $b_{j(i)}^{(+)}$ $(j = 1, \ldots, m)$ the limits, in which $p(y_{(i)}) \geqslant \varepsilon$, where $\varepsilon > 0$ - certain sufficiently small positive number. It is always easy to determine these limits according to the results of computations at the previous step/pitch.

Page 201.

Let us consider now the index of exponential curve in formula (VII.21):

$$\lambda = \frac{\begin{vmatrix} R_{11(l)} & \dots & R_{1m(l)} & a_1 \\ \dots & \dots & \dots & \dots \\ R_{m1(l)} & \dots & R_{nn(l)} & a_n \\ a_1 & \dots & a_n & 0 \end{vmatrix}}{\begin{vmatrix} R_{11(l)} & \dots & R_{1n(l)} \\ \dots & \dots & \dots \\ R_{n1(l)} & \dots & R_{nn(l)} \end{vmatrix}}$$

where

$$a_j = y_{j(l+1)} - y_{j(l)} - F_{j0(l)}\Delta t.$$
$$(j = 1, \dots, n)$$

It is natural to assume that the value of exponential curve is substantially excellent from zero when $|\lambda|<B$, where B - certain constant value, whose values are established/installed from the condition of the proximity to zero of entire integrand in formula (VII.21).

Point $(\mu_{1(l)}, \dots, \mu_{n(l)})$, in vicinity of which $|\lambda|\geqslant B$, it is possible to determine from the condition

$$a_1 = \dots = a_n = 0. \qquad \text{(VII.41)}$$

The borders of this region $(c_{j(i)}^{(-)}, c_{j(i)}^{(+)})$ $(j = 1, \ldots, n)$ can be in the first approximation, found from the formulas

$$\left.\begin{aligned} c_{j(i)}^{(+)} &= y_{j(i+1)} - \Big[F_{j0}(\mu_{1(i)}, \ldots, \mu_{n(i)}) - \\ &- \sum_{l=1}^{m} K_l \,|F_{jl}(\mu_{1(i)}, \ldots, \mu_{n(i)})|\, \sqrt{D[V_l]}\Big]\,\Delta t; \\ c_{j(i)}^{(-)} &= y_{j(i+1)} - \Big[F_{j0}(\mu_{1(i)}, \ldots, \mu_{n(i)}) + \\ &+ \sum_{l=1}^{m} K_l \,|F_{jl}(\mu_{1(i)}, \ldots, \mu_{n(i)})|\, \sqrt{D[V_l]}\Big]\,\Delta t, \end{aligned}\right\} \qquad \text{(VII.42)}$$

where $K_l$ - some constant positive values, which characterize the value of disturbance/perturbation $V_l$ in the standard deviations.

Depending on the form of the function $F_{jl}(y_1, \ldots, y_n)$ values $K_l$ are chosen different. Usually $K_l \geqslant 4$.

Page 202.

If functions $F_{jl}(y_1, \ldots, y_n)$ $(l = 0, 1, \ldots, n)$ very strongly vary with change $y_j (j = 1, \ldots, n)$, then it is possible to do the second approximation/approach, using formulas (VII.42), but during computation $F_{jl}$ instead of $\mu_1, \ldots, \mu_n$ to substitute corresponding values $c_{j(i)}^{(+)}$ or $c_{j(i)}^{(-)}$ $(j = 1, \ldots, n)$, calculated in the first approximation.

Integration in the infinite limits in expression (VII.21) makes sense to replace with integration for the finite domain, in which the

value of integrand is substantially excellent from zero. Integration limits in this case can be determined according to the following formulas:

$$y_{j(i)}^{(-)} = \max\{b_{j(i)}^{(-)}, c_{j(i)}^{(-)}\};$$
$$y_{j(i)}^{(+)} = \min\{b_{j(i)}^{(+)}, c_{j(i)}^{(+)}\}. \qquad \text{(VII.43)}$$

Since $c_{j(i)}^{(-)}$ and $c_{j(i)}^{(+)}$ $(j = 1, ..., n)$ depend on the point, at which the probability density is calculated, then limits $y_{j(i)}^{(-)}$ and $y_{j(i)}^{(+)}$, determined by formula (VII.43), also depend not only on the number of step/pitch (i), but also on point $y_{(i+1)}$. Integral on isolated thus region can be calculated by different methods.

One of the most important questions, which appears during the practical use of a method, is the selection of the method of the representation of multidimensional probability density, i.e. the representation of the function of many variables.

With the low order of the system, when we encounter with the one-dimensional, two-dimensional or three-dimensional probability densities, their representation in the form of tables is possible. In this case one or another the method of interpolation is used. In the simplest cases with the sufficiently low pitch of tables linear interpolation gives completely satisfactory results.

However, the representation of probability density in the form of tables is connected with the known incönveniences, which, first of all, include the need of computing the probability density to (i+1) step/pitch in a large number of points, which substantially increases the time of solution.

Since due to the errors, examined in the previous paragraph, the calculated probability density cannot completely satisfy the conditions of normalization, has sense to introduce into formula (VII.3) normalizing factor. Then

$$p(y_{1(l+1)}, \ldots, y_{n(l+1)}) = \int\limits_{y_{1(l)}^{(-)}}^{y_{1(l)}^{(+)}} \overset{(n)}{\cdots} \int\limits_{y_{n(l)}^{(-)}}^{y_{n(l)}^{(+)}} K_H p(y_{1(l)}, \ldots, y_{n(l)}) \times$$
$$\times p_\Delta(y_{1(l+1)} - y_{1(l)}, \ldots, y_{n(l+1)} - y_{n(l)}, y_{(l)})\, dy_{1(l)} \ldots dy_{n(l)}, \quad \text{(VII.44)}$$

where

$$K_H = \frac{1}{I_{(l)}};$$

$$I_{(l)} = \int\limits_{-\infty}^{\infty} \overset{(n)}{\cdots} \int p(y_{1(l)}, \ldots, y_{n(l)})\, dy_{1(l)} \ldots dy_{n(l)}.$$

The introduction of normalizing factor makes it possible to decrease the accumulation of the error, caused by an inaccuracy in the representation of probability density.

Besides the tabular representation, it is possible to assume the approximation of multidimensional probability density by the segment

of the series/row of resolution in terms of the orthogonal polynomials, as which it is expedient to üse the generalized Chebyshev polynomials - Hermite. Then the probability density of the vector of coordinates can be represented in the form

$$p(y_{1(l)}, \ldots, y_{n(l)}) = Kp^0(y_{1(l)}, \ldots, y_{n(l)})\Big\{b_0 + \sum_{\alpha=1}^{n} b_\alpha H_\alpha + \\ + \sum_{\alpha\beta} b_{\alpha\beta} H_{\alpha\beta} + \sum_{\alpha\beta\gamma} b_{\alpha\beta\gamma} H_{\alpha\beta\gamma} + \sum_{\alpha\beta\gamma\delta} b_{\alpha\beta\gamma\delta} H_{\alpha\beta\gamma\delta} + \cdots\Big\}, \quad \text{(VII.45)}$$

where

$$p^0(y_{1(l)}, \ldots, y_{n(l)}) = \prod_{j=1}^{n} \frac{1}{2\pi} e^{-\frac{z_{j(l)}^2}{2}};$$

$$z_{j(l)} = \frac{y_{j(l)} - \chi_{j(l)}}{\rho_{j(l)}};$$

$$\chi_{j(l)} = \mathrm{M}\left[y_{j(l-1)}\right] + F_{j0}(y_{1(l-1)}, \ldots, y_{n(l-1)});$$

$$\rho_{j(l)} = \frac{b^{(+)}_{j(l-1)} - b^{(-)}_{j(l-1)}}{6};$$

$$H_\alpha = y_{\alpha(l)};$$

$$H_{\alpha\beta} = y_{\alpha(l)} y_{\beta(l)} - a_{\alpha\beta};$$

$$H_{\alpha\beta\gamma} = y_{\alpha(l)} y_{\beta(l)} y_{\gamma(l)} - a_{\alpha\beta} y_{\gamma(l)} - a_{\alpha\gamma} y_{\beta(l)} - a_{\beta\gamma} y_{\alpha(l)};$$

$$H_{\alpha\beta\gamma\delta} = y_{\alpha(l)} y_{\beta(l)} y_{\gamma(l)} y_{\delta(l)} - a_{\alpha\beta} y_{\gamma(l)} y_{\delta(l)} - a_{\alpha\gamma} y_{\beta(l)} y_{\delta(l)} - a_{\alpha\delta} y_{\beta(l)} y_{\gamma(l)} - \\ - a_{\beta\gamma} y_{\alpha(l)} y_{\delta(l)} - a_{\beta\delta} y_{\alpha(l)} y_{\gamma(l)} - a_{\gamma\delta} y_{\alpha(l)} y_{\beta(l)} - a_{\alpha\beta} a_{\gamma\delta} - a_{\alpha\gamma} a_{\beta\delta} - a_{\alpha\delta} a_{\beta\gamma};$$

$$c_{\kappa l} = \begin{cases} 1 \text{ при } \kappa = l, \ (1) \\ 0 \text{ при } \kappa \neq l. \end{cases}$$

$$\kappa, l = \alpha, \beta, \gamma, \delta$$

Key: (1). with.

From the expressions for $H_{\alpha\ldots\delta}$ it is evident that the transfer of

indices $\alpha$, ..., $\delta$ does not affect value $H_{\alpha...\delta}$, therefore the values of coefficients $b_{\alpha...\delta}$ do not vary from the transfer of indices.

Page 204.

A number of different coefficients $b_{\alpha...\delta}$ is determined from the formula

$$N_b = 1 + n + c_n^{2(n)} + c_n^{3(n)} + c_n^{4(n)} + \ldots, \qquad \text{(VII.46)}$$

where $c_n^{\kappa(n)}$ $(\kappa = 2, 3, 4\ldots)$ - number of combinations with the repetitions from the n elements/cells on k;

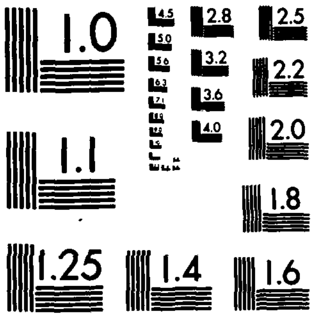
$$c_n^{\kappa(n)} = \frac{(n+\kappa-1)!}{\kappa!\,(n-1)!}.$$

It must be noted that in formula (VII.45) $\chi_{j(i)}$ is not the mathematical expectation of value $y_{j(i)}$, although it is close to it, and $\rho_{j(i)}$ is not standard deviation. Because of this in expansion (VII.45) are present the terms with coefficients $b_0$, $b_\alpha$ and $b_{\alpha\beta}$.

Representation of probability density in the form of expression (VII.45) gives the possibility to reduce the problem to the solution at each step/pitch of the system of linear algebraic equations of $N_b$-order, whose right sides are determined by computing the multiple integrals at $N_b$ points. In this case the algorithm of the solution of the problem of determining the probability density can be presented in the following form.

AD-A143 297 STATIC METHODS IN THE DESIGN OF NONLINEAR AUTOMATIC CONTROL SYSTEMS(U) FOREIGN TECHNOLOGY DIV WRIGHT-PATTERSON AFB OH N I ANDREYEV ET AL. 27 JUN 84 5/9

UNCLASSIFIED FTD-ID(RS)T-1734-83 F/G 12/1 NL




MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

Output data at each step/pitch of solution are coefficients $b_{\alpha\ldots\delta\,(i+1)}$ and $K_{(i+1)}$.

Coefficients $b_{\alpha\ldots\delta(i+1)}$ are defined as the solutions of the following system of linear algebraic equations of $N_b$-order:

$$b_{0(i+1)}+\sum_{\alpha}b_{\alpha(i+1)}H_{\alpha(\nu)}+\sum_{\alpha\beta}b_{\alpha\beta(i+1)}H_{\alpha\beta(\nu)}+\sum_{\alpha\beta\gamma}b_{\alpha\beta\gamma(i+1)}H_{\alpha\beta\gamma(\nu)}+$$
$$+\sum_{\alpha\beta\gamma\delta}b_{\alpha\beta\gamma\delta(i+1)}H_{\alpha\beta\gamma\delta(\nu)}=\frac{p(\nu_{1(i+1)},\ldots,\nu_{m(i+1)})}{p^{0}(\nu_{1(i+1)},\ldots,\nu_{m(i+1)})},\quad \text{(VII.47)}$$

where $\alpha\beta\gamma\delta$; $\alpha\beta\gamma$; $\alpha\beta$ - combination with repetition from m numbers (1, 2, ..., m) on 4, on 3, on 2 respectively; $p^{0}(\nu_{1(i+1)},\ldots,\nu_{m(i+1)})$ - standard probability density at the point $y_{(i+1)}=\nu_{(i+1)}$;

$H_{\alpha\ldots\delta(\nu)}$ - generalized polynomials of Chebyshev - Hermite, which are calculated for $y_{(i+1)}=\nu_{(i+1)}$ from the same formulas, as in expression (VII.45);

$$p(\nu_{1(i+1)},\ldots,\nu_{m(i+1)})=\int\limits_{y_{(i)}^{(-)}}^{y_{(i)}^{(+)}}K_{(i)}B_{(i)}p_{\Delta}(\nu_{(i+1)}-y_{(i)}\,|\,y_{(i)})\,dy_{(i)};$$

$$b_{(i)}=\begin{cases}B^{*}_{(i)} \text{ при } B^{*}_{(i)}>0;\\ 0 \text{ при } B^{*}_{(i)}<0\end{cases} \quad 1$$

$$B^{*}_{(i)}=b_{0(i)}+\sum_{\alpha}b_{\alpha(i)}H_{\alpha(i)}+\sum_{\alpha\beta}b_{\alpha\beta(i)}H_{\alpha\beta(i)}+\sum_{\alpha\beta\gamma}b_{\alpha\beta\gamma(i)}H_{\alpha\beta\gamma(i)}+$$
$$+\sum_{\alpha\beta\gamma\delta}b_{\alpha\beta\gamma\delta(i)}H_{\alpha\beta\gamma\delta(i)}.\quad \text{(VII.48)}$$

Key: (1). with.

Page 205.

Limitations on $B_{(i)}$ it is necessary to input/embed because a finite number of terms of the expansion of probability density is considered, and this can lead to the appearance of negative values low in the absolute value on the borders (here, as before, integral on the vector it indicates multiple integral on its components).

Points $v_{j(i+1)}$ are chosen from interval $[d^{(-)}_{j(i+1)}, d^{(+)}_{j(i+1)}]$:

$$d^{(-)}_{j(i+1)} = b^{(-)}_{j(i)} + F_{j0}(y_{1(i)}, \ldots, y_{n(i)})\Delta t,$$

where

$$y_{j(i)} = b^{(-)}_{j(i)}; \quad \underset{\kappa \neq j}{y_{\kappa(i)}} = \frac{b^{(+)}_{\kappa(i)} + b^{(-)}_{\kappa(i)}}{2};$$

$$d^{(+)}_{j(i+1)} = b^{(+)}_{j(i)} + F_{j0}(y_{1(i)}, \ldots, y_{n(i)})\Delta t,$$

where

$$y_{j(i)} = b^{(+)}_{j(i)}; \quad \underset{\kappa \neq j}{y_{\kappa(i)}} = \frac{b^{(+)}_{\kappa(i)} + b^{(-)}_{\kappa(i)}}{2}.$$

Coefficient $\kappa_{(i+1)}$ is determined from the formula

$$\kappa_{(i+1)} = \frac{1}{I_{(i+1)}}, \qquad \text{(VII.49)}$$

where

$$I_{(l+1)} = \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int B_{(l+1)} dy_{(l+1)};$$

$$B_{(l+1)} = b_{0(l+1)} + \sum_{\alpha} b_{\alpha(l+1)} H_{\alpha(l+1)} + \sum_{\alpha\beta} b_{\alpha\beta(l+1)} H_{\alpha\beta(l+1)} + \\ + \sum_{\alpha\beta\gamma} b_{\alpha\beta\gamma(l+1)} H_{\alpha\beta\gamma(l+1)} + \sum_{\alpha\beta\gamma\delta} b_{\alpha\beta\gamma\delta(l+1)} H_{\alpha\beta\gamma\delta(l+1)}.$$

It must be noted that values $\kappa_{(l+1)}$ here, as a rule, knowingly differ from one.

6. Task about the probability of the nonappearance of the coordinates of system for the limitations.

In a series/row of applied problems it is necessary to solve the problems, connected with the determination of the probability of reaching/achievement by certain function of the preset borders. The determination of the probability of the fact that in the interval of time $[t_0, T]$ the preset limitations to the coordinates, are observed is the simplest task of this type, i.e. the inequality

$$h^{(-)} < Y(t) < h^{(+)}.$$

is fulfilled.

The approximate method, described in present chapter, makes it possible to solve this task.

Page 206.

Let us record expression for random vector $Y_{(i+1)}$ at moment/torque $t_i + \Delta t$ through its values $Y_{(i)}$

$$Y_{(i+1)} = Y_{(i)} + \Delta Y_{(i)}.$$

It is analogous how this is done during the use of theory of Markov processes [91], let us introduce into the examination the probability density $p^*(y_{(i)})$ of the fact that at moment/torque $t_i$ random vector Y will be found in interval $[Y_{(i)}, Y_{(i)} + dY_{(i)}]$ when for time $t_i - t_0$ vector Y not fall outside the limits of interval $[h^{(-)}, h^{(+)}]$.

Task consists of the determination of the probability of the fact that in the interval of time $[t_0, t_N = T]$ will be implemented inequality $h^{(-)} \leq Y < h^{(+)}$ if it is known that the probability density of vector Y at moment/torque $t_0$ is equal to $p(y_0)$ and

$$\int_{h^{(-)}}^{h^{(+)}} p(y_{(0)})\, dy_{(0)} = 1.$$

The probability density of vector Y at moment/torque $t_1 = t_0 + \Delta t$ when for the time $\Delta t = t_1 - t_0$ vector Y will not leave for the limitations, can be determined according to the formula

$$p^*(y_{(1)}) = \int_{h^{(-)}}^{h^{(+)}} p(y_{(0)})\, p_\Delta(y_{(1)} - y_{(0)}/y_{(0)})\, dy_{(0)}. \quad \text{(VII.50)}$$

Here, as before, $p_\Delta(y_{(1)} - y_{(0)}/y_{(0)})$ - the conditional probability density of the vector of increments, and integral on the vector should be considered as multiple integral on its components.

Let us consider moment/torque $t_2=t_1 + \Delta t$ further and it is determined the probability density of vector $Y_{(2)}$ under the condition of the nonappearance of vector Y in the interval of time $t_2 - t_0$ beyond the limits of the preset borders:

$$p^*(y_{(2)}) = \int_{h^{(-)}}^{h^{(+)}} p^*(y_{(1)})\, p_\Delta(y_{(2)} - y_{(1)}/y_{(1)})\, dy_{(1)}.$$

it is analogous for any i moment

$$p^*(y_{(i)}) = \int_{h^{(-)}}^{h^{(+)}} p^*(y_{(i-1)})\, p_\Delta(y_{(i)} - y_{(i-1)}/y_{(i-1)})\, dy_{(i-1)}, \quad \text{(VII.51)}$$

where $p^*(y_{(i)})$ and $p^*(y_{(i-1)})$ - probability density of vector Y at moments/torques $t_i$ and $t_{i-1}$ when outside time $t_i - t_0$ and $t_{i-1} - t_0$ respectively vector Y does not fall for the limitations.

Page 207.

At moment/torque $t_N = T$ we will obtain

$$p^*(y_{(N)}) = \int_{h^{(-)}}^{h^{(+)}} p^*(y_{(N-1)})\, p_\Delta(y_N - y_{(N-1)}/y_{(N-1)})\, dy_{(N-1)}. \quad \text{(VII.52)}$$

Then probability that for time $T-t_0$ vector Y not fall outside the limits of limitations, is determined by the following formula:

$$P(y, T) = \int_{h^{(-)}}^{h^{(+)}} p^*(y_{(N)})\,dy_{(N)}. \qquad \text{(VII.53)}$$

Thus, the algorithm of the determination of probability density $p^*(y_{(i)})$ coincides with the algorithm of the determination of the probability density of the vector of output coordinates, presented in the previous paragraphs, if we instead of infinite integration limits take the limitations, superimposed on the values of vector Y. In this case it is necessary to take into account some special features/peculiarities.

If we use representation $p^*(y)$ in the form of tables, then the always normalizing factor in formula (VII.44) $K_{ii} = 1$, since $I_{(i)}$ will differ from 1, if somewhere to moment/torque $t_i$ vector Y exceeds the limits of the preset borders.

For the same reasons in formula (VII.49) it is necessary to take $K_{(i+1)} = 1$.

During the determination of the range of integration of equality (VII.43) it is necessary to replace with the following:

$$y^{(-)}_{j(i)} = \max\{b^{(-)}_{j(i)},\ c^{(-)}_{j(i)},\ h^{(-)}_{(j)}\};$$
$$y^{(+)}_{j(i)} = \min\{b^{(+)}_{j(i)},\ c^{(+)}_{j(i)},\ h^{(+)}_{j}\}. \qquad \text{(VII.54)}$$

Example. Let us consider based on the example of the analysis of the behavior of the automatic control system the use of a method, presented in present chapter, described by the nonlinear differential equations of the form

$$\left.\begin{aligned} \frac{dY_1}{dt} &= b_1(Y_1, Y_2) + (a_1 + \Delta a_1)\cos Y_2 + [(a_2 + \Delta a_2) + (a_3 + \Delta a_3) + \\ &\quad + (a_4 + \Delta a_4)\, Y_2]\, Y_1^2; \\ \frac{dY_2}{dt} &= b_2(Y_1 Y_2) - (a_1 + \Delta a_1)\frac{\sin Y_2}{Y_1} + k[(a_3 + \Delta a_3) + \\ &\quad + (a_4 + \Delta a_4)\, Y_2]\, Y_1, \end{aligned}\right| \qquad \text{(VII.55)}$$

where $b_1(Y_1, Y_2)$ and $b_2(Y_1, Y_2)$ - some preset nonrandom functions of coordinates $Y_1$ and $Y_2$ (in the general case nonlinear);

$\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ - determined parameters;

$\Delta\alpha_1$, $\Delta\alpha_2$, $\Delta\alpha_3$, $\Delta\alpha_4$ - independent central probable deviations of the parameters, distributed according to the normal law with the known dispersions.

Page 208.

It is necessary to determine the probability density of coordinates $Y_1$ and $Y_2$ at the fixed/recorded points on the interval $[t_0, T]$.

Let us introduce the designations

$$\Delta a_1 = V_1;\ \Delta a_2 = V_2;\ \Delta a_3 = V_3;\ \Delta a_4 = V_4;$$

$D[V_i]$ $(i = 1, \ldots, 4)$ th dispersion of probable deviations.

Let us lead system (VII.55) to form (VII.17).

We will obtain

$$\frac{dY_1}{dt} = F_{10} + F_{11}V_1 + F_{12}V_1 + F_{13}V_3 + F_{14}V_4;$$

$$\frac{dY_2}{dt} = F_{20} + F_{21}V_1 + F_{23}V_3 + F_{24}V_4,$$

where

$$F_{10} = b_1(Y_1, Y_2) + a_1 \cos Y_2 + (u_2 + a_3 + a_4 Y_2) Y_1^2;$$

$$F_{11} = \cos Y_2;\ F_{12} = Y_1^2;\ F_{13} = Y_1^2;\ F_{14} = Y_2 Y_1^2;$$

$$F_{20} = b_2(Y_1, Y_2) + F_{21}u_1 + F_{23}u_3 + F_{24}u_4;$$

$$F_{21} = -\frac{\sin Y_2}{Y_1};\ F_{23} = kY_1;\ F_{24} = kY_1Y_2.$$

Page 209.

For the solution of problem we will use formula (VII.21), after replacing in it infinite limits by the final ones according to formula (VII.43):

$$p(y_{1(i+1)}, y_{2(i+1)}) = \frac{1}{2\pi} \int\limits_{y_{1(i)}^{(-)}}^{y_{1(i)}^{(+)}} \int\limits_{y_{2(i)}^{(-)}}^{y_{2(i)}^{(+)}} \frac{p(y_{1(i)}, y_{2(i)})}{\sqrt{|R_{(i)}|}} e^{\frac{\lambda}{2}} dy_{1(i)} dy_{2(i)},$$

where

$$\lambda = \frac{\begin{vmatrix} R_{11(i)} & R_{12(i)} & \alpha_1 \\ R_{21(i)} & R_{22(i)} & \alpha_2 \\ \alpha_1 & \alpha_2 & 0 \end{vmatrix}}{|R_{(i)}|};$$

$$\alpha_1 = y_{1(i+1)} - y_{1(i)} - F_{(10i)}\Delta t;$$

$$\alpha_2 = y_{2(i+1)} - y_{2(i)} - F_{20(i)}\Delta t;$$

$$|R_{(i)}| = \begin{vmatrix} R_{11(i)} & R_{12(i)} \\ R_{21(i)} & R_{22(i)} \end{vmatrix};$$

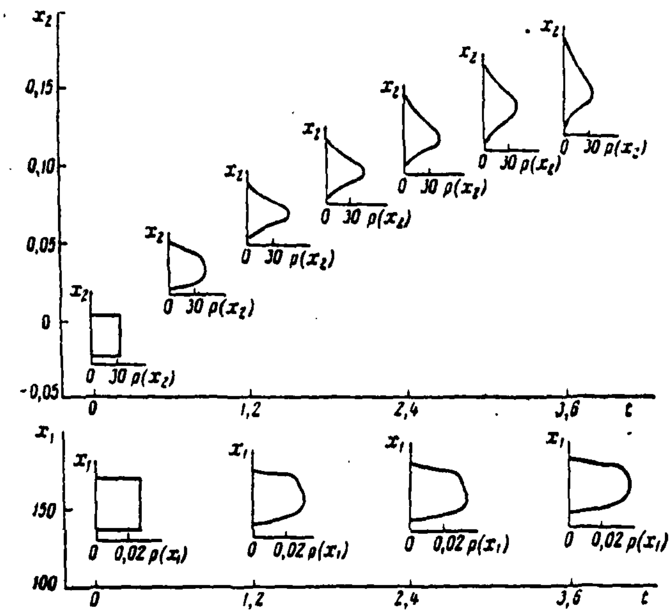
$$\underset{(\kappa,\, j=1,\, 2)}{R_{\kappa j(i)}} = \Delta t^2 \sum_{l=1}^{4} F_{jl(i)} F_{\kappa l(i)} D[V_l];$$

$$\underset{(j=1,\, 2)}{y_{j(i)}^{(-)}} = \max\{b_{j(i)}^{(-)}, c_{j(i)}^{(-)}\};$$

$$\underset{(j=1,\, 2)}{y_{j(i)}^{(+)}} = \min\{b_{j(i)}^{(+)}, c_{j(i)}^{(+)}\};$$

$$c_{j(i)}^{(-)} = y_{j(i+1)} - \left[F_{j0}(\mu_1, \mu_2) + \sum_{l=1}^{4} K_l \mid F_{jl}(\mu_1 \mu_2) \mid \sqrt{D[V_l]}\right] \Delta t;$$

$$c_{j(i)}^{(+)} = y_{j(i+1)} - \left[F_{j0}(\mu_1 \mu_2) - \sum_{l=1}^{4} K_l \mid F_{jl}(\mu_1 \mu_2) \mid \sqrt{D[V_l]}\right] \Delta t.$$



Graphs of the probability density of coordinates $Y_1$ and $Y_2$.

Values $\mu_1$ and $\mu_2$ are determined here on the following iterative cycle:

for the n approximation/approach

$$\mu_1^{(N)} = y_{1(i+1)} - F_{10}(\mu_1^{(N-1)}, \mu_2^{(N-1)})\Delta t;$$
$$\mu_2^{(N)} = y_{2(i+1)} - F_{20}(\mu_1^{(N-1)}, \mu_2^{(N-1)})\Delta t;$$
$$\alpha_1^{(N)} = y_{1(i+1)} - \mu_1^{(N)} - F_{10}(\mu_1^{(N)}, \mu_2^{(N)})\Delta t;$$
$$\alpha_2^{(N)} = y_{2(i+1)} - \mu_2^{(N)} - F_{20}(\mu_1^{(N)}, \mu_2^{(N)})\Delta t.$$

One should stop at that approximation/approach, for which $\alpha_1^{(N)} < \varepsilon_1$; $\alpha_2^{(N)} < \varepsilon_2$.

Let us note that iterative cycle proposed above descends not with any form of the function $F_{j0}(y_1, y_2)\Delta t$ $(j = 1, 2)$.

Page 210.

As $b_{j(i)}$ $(j = 1, \ldots, n)$ are accepted such limiting values $y_{j(i)}$, on leaving for which $p(y_{(i)}) \approx 0$.

The unknown probability density of components is determined from the formulas

$$p(y_{1(t)}) = \int_{-\infty}^{\infty} p(y_{1(t)}, y_{2(t)})\, dy_{2(t)},$$

$$p(y_{2(t)}) = \int_{-\infty}^{\infty} p(y_{1(t)}, y_{2(t)})\, dy_{1(t)}.$$

Fig. VII.1 gives the graphs of the probability density of coordinates $Y_1$ and $Y_2$ in the section t=0-3.6 s, obtained during the use of the algorithm described above for the following numerical values of the parameters:

$$b_1(Y_1, Y_2) = -\left[2{,}6 + Y_2 + \left(0{,}9\cdot 10^{-6} + 0{,}7\cdot 10^{-3} Y_2^2\right) Y_1^2\right];$$

$$b_2(Y_1, Y_2) = \frac{1}{Y_1}(9{,}5 + 2{,}5 Y_2);$$

$$\varkappa = 2{,}3;\ a_1 = 5{,}3;\ a_2 = 0{,}1\cdot 10^{-4};$$

$$a_3 = -0{,}3\cdot 10^{-4};\ a_4 = 0{,}8\cdot 10^{-3};$$

$$D[V_1] = 0{,}01;\quad D[V_2] = 0{,}4\cdot 10^{-13};$$

$$D[V_3] = 0{,}6\cdot 10^{-11};\ D[V_4] = 0{,}5\cdot 10^{-8}.$$

---- --

Page 211.

# Chapter VIII.

## SOME TASKS OF THE SYNTHESIS OF NONLINEAR CONTROL SYSTEMS UNDER RANDOM INFLUENCES.

### 1. Formulation of the problem of synthesis.

The connection/communication between the input information and the control pressure is determined by the algorithm of the control system. With the optimum algorithm of the control system is realized optimal control of movement when the outer limit of the previously selected criterion of optimality attains in the process of motion and the conditions for limitations are satisfied.

The task of the synthesis of optimal steering of the controlled objects can be divided into three stages: the synthesis of the optimum program of control; the synthesis of the optimum law of control; the synthesis of the control system, which realizes optimal

steering.

The program of control is a change in the control pressure for guaranteeing the required optimum motion. The law of control reflects the process of molding of the control pressure according to the input information for the ideal system of control (ideal regulator).

Page 212.

The nonlinear laws of control acquire an increasing importance. In the general case even the linear optimum laws of control upon consideration of limitations are nonlinear. To the nonlinear laws they lead different contemporary methods of optimization.

During the equipment realization of the selected law of control the obtained equation of the control system can differ significantly from the equation of the law of control. Difference can consist not only in the degree of differential equation, which, as a rule, is raised. Taking into account of the objective parameters and parameters of single elements/cells and devices/equipment the equation of motion of the control system usually is converted into the nonlinear, frequently with the variable parameters. The internal random disturbances, connected with the work of the single elements of system, can appear. In the implementation of the law of the

control digital controller the system is converted into the discrete/digital. Digital computer with the work in the control loop also can be the source of random disturbances as a result of the quantization of control signal both on the time and in the amplitude.

A difference in the real equation of the system of control from the selected optimum law of control can cause the essential divergence of the real guided motion from the optimum and lead to the motion, whose character does not correspond to the given one or is unstable.

Thus, the real equations of system of control can strongly differ from the law of control, and the full/total/complete control system with the controlled object - present considerably more complex system described by nonlinear differential equations. Therefore the synthesis of the program and law of control in the task of optimal steering is only preliminary stage.

The final synthesis of steering is accomplished in the third stage, when are optimized the parameters of the control system, the parameters of all or some elements/cells. To these parameters can relate the factors of amplification, the time constants, limitation of characteristics, the parameters digital of controller, some design parameters of elements/cells, etc., i.e., all those parameters, which

characterize the operator of meters and the operator of the amplifying and converting device/equipment with the actuating controls of the control system.

During the optimization different criteria of optimality can be accepted. To join all requirements for the control system is difficult in one criterion. Therefore the principal criteria, which determine the fundamental required properties of system, are chosen, and on other properties assign the specific limitations.

Page 213.

As the main criteria it is possible, for example, to take the criterion of the accuracy of the endurance/seasoning the preset program motion or the criterion of the minimum of energy consumption per control. Latter/last criterion appears when the program of control is not optimized, for example in the case of the simple maintenance of the constant value of the controlled (adjustable) value.

Since only the law of control can ensure the required accuracy of the motion of system, as the criterion it is necessary to examine the error of the endurance/seasoning the program of control.

The controlled object together with the control system is subjected to random interactions. Therefore the probabilistic characteristics of the error of system must be examined as criteria for the selection of the law of control. Actually, the initial conditions of moving the object, as a rule, are by chance, the exciting accelerations are the random functions of time. Therefore the divergences of the finite values of the parameters of motion are also random.

The fullest/most total/most complete statistical criterion is the "maximum of the probability" of the fact that the parameters of motion will be found within the preset limits $|C_1$ and $C_2|$, i.e., the error of the system E is located in the specific borders:

$$P = P(C_1 \leqslant E \leqslant C_2) \qquad \text{(VIII.1)}$$

or

$$P = P(E \leqslant C). \qquad \text{(VIII.2)}$$

The optimization of the control system must be carried out through the maximum value of this probability. The characteristics of the system of control is determined by functional from output coordinates $\Phi_i$. The function, called in the mathematical set theory characteristic function $X_i(\Phi_i)$. determines the probabilistic characteristic of functional With satisfaction of condition (VIII.1)

$$X_i = \left[\frac{1}{2}\left(1 - \frac{\Phi_i - C_2}{|\Phi_i - C_2|}\right)\right]\left[\frac{1}{2}\left(1 + \frac{\Phi_i - C_1}{|\Phi_i - C_1|}\right)\right]. \qquad \text{(VIII.3)}$$

and for limitation (VIII.2)

$$X_i = \frac{1}{2}\left(1 - \frac{\Phi_i - C}{|\Phi_i - C|}\right). \qquad \text{(VIII.4)}$$

It is easy to note that characteristic function $X_i$ is equal to 1, if $\Phi_i = E$ satisfy conditions (VIII.1) or (VIII.2), and $X_i = 0$ otherwise.

In most general view the formulation of the problem of synthesis indicates the following.

Page 214.

Let the projected control system together with the object of control be described by the system of nonlinear differential equations in the vector form

$$\frac{dY}{dt} = R(Y, K, Z, t), \qquad \text{(VIII.5)}$$

where R - nonlinear operator;

Y - n-dimensional vector of coordinates $Y_i$ of unit and control system;

K - the r-dimensional vector of parameters $K_j$ of the system of control, whose selection is necessary to carry out;

Z - vector random function, which reflects external and internal

random interactions;

t - current time.

Vector K must ensure the extremum of certain criterion I, which depends on the operators of the system and its parameters.

Thus, the task of the synthesis of statistically optimum system can be represented as the search for the extremum of the function of many independent variables.

As is known, the solution of this problem by analytical methods is possible only for the simplest cases. It is possible to solve this task, applying different approximate numerical methods with the use of electronic computers. Their efficiency is determined by the selected machine algorithm, while the efficiency of algorithm, in turn, is determined in by the bulk of necessary solutions of reference system. The known methods of solving the nonlinear stochastic equations assume the presence of the random interactions, preset mainly not in the form of random functions, but in the form of the determined functions of time and random variables. Therefore for solving stated problem it is necessary from system (VIII.5) to switch over to the system

$$\frac{dY}{dt} = R(Y,\ K,\ V,\ t), \qquad \text{(VIII.6)}$$

where V - vector of the random variable, constructed according to the vector random function Z(t).

It is further necessary to define the value of criterion I as the functions of the form

$$I = \sum_i X_i B_i, \qquad \text{(VIII.7)}$$

where $X_i$ — certain characteristic function, determined by criterion by the I and preset limitations;

$B_i$ — weight coefficients.

Page 215.

In connection with the fact that there are no general methods of determining the nonlinear law of control for the solution of the problem of the statistical synthesis of complex nonlinear systems, usually for linearized preset control components and controlled object is determined the approximate law of control, i.e., the approximate structure of the control system.

Further, according to the preset nonlinear characteristics of the object of control, to substantially nonlinear elements/cells and to the converting devices/equipment of system, and also according to the preliminarily selected law of control is comprised the structural

scheme of the control system and the equations of its motion are written/recorded.

Then the random variables, which characterize these disturbances/perturbations, are determined according to the given statistical characteristics of random disturbances.

Now it is necessary to select the coefficients of the law of control in such a way by the method of statistical synthesis that the probability of falling of the output coordinates of the object of control in the preset region would achieve its greatest value. With selected thus coefficients of the law of control we will obtain the optimum law of control.

During the equipment realization of the optimum law of control the new design parameters (coefficients) of control components appear. They include the factors of amplification (transmission), the time constants, time lag, dead zone, limitation, the coefficients of logical units, etc.

In the mathematical description of this real system of control with the real object of control the system of controls can differ significantly from the equations obtained earlier.

The task of the synthesis of the optimum control system is the determination of such values of the design parameters (coefficients) of the elements of system, with which the probability mentioned above reaches the greatest value with the stabilization of a control process.

We will count control process of stable, if the standard deviations of the initial parameters of motion, design parameters of the system of control, external and internal interactions from their selected values cause the standard deviations of the output coordinates of the controlled object.

Experiment of the synthesis of the complex control systems shows that the stages of the solution of problem must be the following:

1. Finding the design parameters of the control system, which ensure the minimum of the divergence of its output coordinates at the condition for the stabilization of a control process without taking into account random interactions;

2. Determination of the design parameters of the control system under the random influences on the criterion of the maximum of the probability of satisfaction to the superimposed limitations.

Page 216.

3. Refinement of values of design parameters of control system by its partial or full/total/complete simulation in computer. During the partial simulation the motion of the controllable object is simulated in the computer, to which is connected the real control system. The controlling and exciting interactions, close to the real ones, are input/embedded in model and equipment for control.

The probabilistic characteristics of the coordinates of the system of control and object are usually the specific values, determined as a result of simulation.

For determining these characteristics it is important to have a method, which did not require the transformations of initial system of differential equations, since only in this case it is possible to apply the models, which contain, besides computers, also real equipment of the control systems. Such methods, besides ordinary statistical simulation, include also the methods of equivalent disturbances/perturbations, interpolation polynomials and statistical nodes, presented respectively in chapter IV, V and VI.

The synthesis of the nonlinear law of control can go along two paths. The first path assumes finding the approximate form of the law

of control via the consecutive sorting/excess of its possible components. Along the alternate path first it is determined the law of the control of simplified system of one of the known methods, and then the coefficients of the law of control of the method of statistical optimization.

The coordinates of system, which are the functional of the form

$$\Phi_i = \Phi_i(K_1, K_2, \ldots, K_s, V_1, V_2, \ldots, V_m, Y_0, t_0, t_\kappa), \quad i = 1, 2, \ldots, p. \qquad \text{(VIII.8)}$$

are the criteria of the optimality of the law of control.

Under the known initial conditions $Y_0$, $t_0$ the value of each functional $\Phi_i$ will depend on time $t_\kappa$, random interactions $V_1, V_2, \ldots, V_m$ and parameters (coefficients) of the law of control $K_1, K_2, \ldots, K_s$. For each moment of time the functional $\Phi_i$ will be the function of variables $V_m, K_s$:

$$\Phi_i = \Phi_i(K_1, K_2, \ldots, K_s, V_1, V_2, \ldots, V_m). \qquad \text{(VIII.9)}$$

The characteristic function $\Phi_i$

$$X_i\{\Phi_i(K_1, K_2, \ldots, K_s, V_1, V_2, \ldots, V_m)\}. \qquad \text{(VIII.10)}$$

is the probabilistic characteristic of functional.

For determining the probabilistic system characteristics it is necessary to determine the function

$$F = M\{X_i\{\Phi_i(K_1, K_2, \ldots, K_s, V_1, V_2, \ldots, V_m)\}\}. \qquad \text{(VIII.11)}$$

Page 217.

Since the mathematical expectation of functional in the random variables is the functions only of nonrandom values $K_s$, then we will obtain

$$F = M[X_l\{\Phi_l(K_s)\}] = F(K_1, K_2, \ldots, K_s). \qquad \text{(VIII.12)}$$

Thus, value F, which determines the probability of the finding of the output parameters in the preset region, is the function of the coefficients of the law of control.

It is obvious that the minimum value of function F will correspond to optimum system on the criterion accepted, and selected coefficients $K_s = K_s^*$ will be optimum, if

$$F(K_s^*) = \min F(K_s), \qquad \text{(VIII.13)}$$

in the permissible range of values of the coefficients of the law of control.

Consequently, the task of finding the optimum coefficients of the law of control is reduced to the task of determining the smallest value of the function of many variables.

On the other hand, by assuming that the computation of mathematical expectation is realized according to the method of statistical nodes (Chapter VI), it is possible taking into account equality (VIII.4) to represent expression for $M[X_i(\Phi_i)]$ in the form

$$M = \sum_{\kappa_1, \kappa_2, \ldots, \kappa_m} \frac{1}{2}\left[1 - \frac{\Phi_i(t, K_s \mu_{m\kappa_m}) - C}{\left|\Phi_i(t, K_s, \mu_{m\kappa_m}) - C\right|}\right] \prod_{j=1}^{m} \rho_{j\kappa_j}. \quad (VIII.14)$$

The method of consecutive variations in the coefficients, when system is solved for all possible combinations of the parameters, is the propagated method of the search for the optimum combinations of coefficients. Criterion value of quality is determined for each solution and are chosen such coefficients, with which is obtained the maximum of probability.

This method requires the enormous volume of computational works even for a small number of optimizable coefficients. Therefore method is used in the approximate version, determining the criterion of the quality of system for single variations in the coefficients, that it does not give guarantee in the optimality of the selected values of coefficients. For the solution of this problem it is possible to use the modified method of gradient or a method of random directions.

With the synthesis of optimum nonlinear system of control the criterion of the optimality of the system of control (as with the synthesis of the law of control) is the functional

$$\Phi_i = \Phi_i(K_r, V_m, Y_0, t_0, t_k), \qquad \text{(VIII.15)}$$

where $K_r$ — the design parameters of the control system, which are subject to selection.

Page 218.

Now the function

$$F = M[X_i[\Phi_i(K_r)]] = F(K_1, \ldots K_r). \qquad \text{(VIII.16)}$$

With the synthesis of the control system, in comparison with the synthesis of the law of control, are complicated equations, that describe dynamics system, a number of optimizable parameters increases $K_r$.

2. Directed search for the optimum parameters of the control system.

The directed search for the design parameters of automatic control system can be realized by methods of gradient and fastest descent.

The algorithm of the method of gradient in general form can be

represented as follows.

Let

$$H = \mathrm{grad}\, F_0(K^0)$$

be the vector of the gradient of function $F_0(K)$, calculated at point $K^0$. Let us select the new point $K^1$, distant behind point $K^0$ at a distance $\epsilon>0$ in the direction, for the reciprocal vector of gradient, i.e.

$$K^{(1)} = K^0 - \varepsilon H.$$

Value $\epsilon$ is called the step.

If the values of function $F_0$, calculated at points $K^0$ and $K^{(1)}$ satisfy the inequality

$$F_0(K^{(1)}) < F_0(K^0),$$

then the transition to the following point $K^2$ according to the formula

$$K^{(2)} = K^{(1)} - \varepsilon H.$$

is realized.

Value $\epsilon$ is reduced in the opposite case, and value $F_0(K^1)$ again is calculated and it is compared with $F_0(K^0)$.

The method of successive approximations, which consists of the

following, is based on this principle.

The arbitrary vector

$$K^{(1)}=(K_1^1,\ K_2^1,\ \ldots,\ K_r^1)$$

as the first approximation of the unknown vector $K^0$ initially is chosen.

Page 219.

It is possible to recommend the value of the vector of the parameters of the control system to choose so that the minimum value of error in the absence of random disturbances would be provided.

Further is determined the direction

$$H^{(1)}=(H_1^{(1)},\ H_2^{(1)},\ \ldots,\ H_r^{(1)}),$$

the approximately equal to gradient on K from function $F_0(K)$ with $K=K^1$.

The second approximation/approach

$$K^{(2)}=(K_1^{(2)},\ K_2^{(2)},\ \ldots,\ K_r^{(2)})$$

is determined from the formula

$$K_i^{(2)}=K_i^{(1)}-\varepsilon^{(1)}H_i^{(1)},\quad i=1,\ 2,\ \ldots,\ r, \qquad \text{(VIII.17)}$$

where the step/pitch $\varepsilon^1$ is chosen from the condition

$$F_0(K^{(2)}) < F_0(K^{(1)}).$$

Continuing by the analogous shape of computation, we obtain the sequence of vectors $K^{(1)}, K^{(2)}, \ldots, K^{(n)}$, with which are satisfied the conditions

$$F_0(K^{(1)}) > F_0(K^{(2)}) > \ldots \cdot F_0(K^{(n-1)}) > F_0(K^{(n)}). \quad \text{(VIII.18)}$$

After the determination of first approximation $K^1$, and also $F_0(K^1)$ values $H_i^{(1)}$ are determined as a result of computing the functional $F_0(K_1^{(1)}, \ldots, K_{i-1}^{(1)}, K_i^{(1)} + \Delta_i, K_{i+1}^{(1)}, \ldots K_r^{(1)})$ in the following form:

$$H_i^{(1)} = \frac{1}{\Delta_i}\left[F_0(K_1^{(1)}, \ldots, K_{i-1}^{(1)}, K_i^{(1)} + \Delta_i, K_{i+1}^{(1)}, \ldots, K_r^{(1)}) - \right.$$
$$\left. - F_0(K_1^{(1)}, \ldots, K_{i-1}^{(1)}, K_i^{(1)}, K_{i+1}^{(1)}, \ldots, K_r^{(1)}) \right]. \quad \text{(VIII.19)}$$

In expression (VIII.19) value $\Delta_i$ — a "small" change of the i parameter of initial system (i=1, 2, ..., r).

The value of step/pitch $\varepsilon^{(1)} \geq 0$ can be variable, and it can be determined by the method of division in half.

In the latter case is taken an arbitrary "small" number $\varepsilon^1{}_1 > 0$ and it is determined

$$K_i^{(2)} = K_i^{(1)} - \varepsilon_1^{(1)} H_i^{(1)}, \; i = 1, 2, \ldots, r \quad \text{(VIII.20)}$$

and

$$F_0(K_1^{(2)}).$$

where

$$K_1^{(2)} = (K_{11}, K_{21}, \ldots, K_{r1}).$$

With fulfilling of the inequality

$$F_0(K_1^{(2)}) < F_0(K^{(1)}) \qquad \text{(VIII.21)}$$

it is accepted $\varepsilon^1 = \varepsilon^1_1$.

Page 220.

In the case of the nonobservance of condition (VIII.21) it is accepted

$$\varepsilon_2^{(1)} = \frac{1}{2}\varepsilon_1^{(1)}. \qquad \text{(VIII.22)}$$

Step $\varepsilon^1_1$ again is made more precise.

If

$$\varepsilon_3^{(1)} = \frac{1}{2}\varepsilon_2^{(1)},$$

then refinement before obtaining of the inequality

$$F_0(K_3^{(2)}) < F_0(K^{(1)}),$$

where $$K_{i\nu}^{(2)} = K_i^{(1)} - \varepsilon_\nu^{(1)} H_i^{(1)}, \quad i = 1, 2, \ldots, r,$$

and $$K_\nu^{(2)} = (K_{1\nu}, K_{2\nu}, \ldots, K_{r\nu}),$$

$\nu \geq 1$ - certain whole number, is continued;

$$\varepsilon_{\nu}^{(1)} = \frac{\varepsilon_{1}^{(1)}}{2^{\nu-1}}.$$

Thus are found vectors $K^{(1)}, K^{(2)}, \ldots, K^{(s)}$, which ensure fulfilling of the inequalities

$$F_0(K^{(1)}) > F_0(K^{(2)}) > \ldots > F_0(K^{(s)}).$$

For determining the coordinates of vector $K^{(s+1)}$ it is necessary to produce computations according to the formula

$$K_i^{(s+1)} = K_i^{(s)} - \varepsilon^{(s)} H_i^{(s)}, \quad i = 1, 2, \ldots r. \qquad \text{(VIII.23)}$$

The value of step/pitch $\varepsilon_1^{(s)} \geqslant 0$ is determined analogously $\epsilon^1$. Direction $H^{(s)} = (H_1^{(s)}, H_2^{(s)}, \ldots, H_r^{(s)})$ is determined from the following of the dependences:

$$H_j^{(s)} = \sigma_0 \alpha_{0j}(K^{(s)}) + \sum_{i \in I} \sigma_i \alpha_{ij}(K^{(s)}), \quad j = 1, \ldots, r; \qquad \text{(VIII.24)}$$

here I - many indices i=1, 2, ..., $l$, for which are made the equalities

$$F_i(K^{(s)}) = 0; \qquad \text{(VIII.25)}$$

$$\alpha_i = (\alpha_{i1}, \ldots, \alpha_{ir});$$

$$\alpha_{0j}(K^{(s)}) = \frac{1}{\Delta_j}\left[F_0(K_1^{(s)}, \ldots, K_{j-1}^{(s)}, K_j^{(s)} + \Delta_j, K_{j+1}^{(s)}, \ldots, K_r^{(s)}) - \right.$$
$$\left. - F_0(K_1^{(s)}, \ldots, K_{j-1}^{(s)}, K_j^{(s)}, K_{j+1}^{(s)}, \ldots, K_r^{(s)})\right];$$

$$\alpha_{ij}(K^{(s)}) = \frac{1}{\Delta_j}\left[F_i(K_1^{(s)}, \ldots, K_{j-1}^{(s)}, K_j^{(s)} + \Delta_j, K_{j+1}^{(s)}, \ldots, K_r^{(s)}) - \right.$$
$$\left. - F_i(K_1^{(s)}, \ldots, K_{j-1}^{(s)}, K_j^{(s)}, K_{j+1}^{(s)}, \ldots, K_r^{(s)})\right].$$

Page 221.

Vector $\alpha_i\,(K^{(s)})$ is approximately equal to gradient on K for $F_i(K)$, when $K = K^{(s)}$; coefficients $\sigma_i \geqslant 0$ $(i = 0, 1, \ldots, l)$ are determined from the condition that direction $(-H^{(s)})$ does not leave from region $\Omega_K$, which is determined by the limitations

$$F_i(K) \leqslant 0 \quad (i = 1, 2, \ldots, l).$$

The process of the determination of design parameters $K^\circ = (K_1^0, \ldots, K_r^0)$ is considered completed, when

$$K_j^{(0)} \quad K_j^{(s)} \quad (j \quad 1, \ldots, r)$$

and with the preset accuracy is fulfilled the inequality

$$\sum_{j=1}^{r} \alpha_{0j}(K^{(s)}) H_j^{(s)} < 0. \qquad \text{(VIII.26)}$$

With satisfaction of condition (VIII.26) $K^{(s+1)}$ it is determined from the expression

$$K^{(s+1)} = K^{(s)} - \varepsilon^{(s)} H^{(s)}.$$

Value $\varepsilon^{(s)} \geqslant 0$ is chosen from the following conditions:

$$F_i(K^{(s+1)}) \leqslant 0 \quad (i = 1, \ldots, l)$$
$$F_0(K^{(s+1)}) < F_0(K^{(s)}).$$

Coefficients $\sigma_i \geqslant 0$ $(i = 0, 1, \ldots, l)$ are chosen from the condition of obtaining the greatest value

$$\sum_{l=1}^{r} \alpha_{0l}(K^{(s)}) H_l^{(s)}.$$

Let us consider the simplest computational algorithm of gradient method.

Let us assume that in the given one to system it is necessary to select design parameters $K_1, K_2, ..., K_r$ such that the output coordinate of object, for example $Y_1$, would be within the limits

$$0 < Y_1 \leqslant C_1. \qquad \text{(VIII.27)}$$

Consequently, it is necessary to obtain this sequence of the managers of the vectors

$$K^{(0)}, K^{(1)}, \ldots, K^{(s)},$$

where

$$K = (K_1, K_2, \ldots, K_r),$$

so that would be fulfilled the relationships/ratios

$$Y_1(K_1^{(i+1)}, K_2^{(i+1)}, \ldots, K_r^{(i+1)}, t) < Y_1(K_1^{(i)}, K_2^{(i)}, \ldots, K_r^{(i)}, t).$$

Page 222.

For the optimum transition/junction from one vector $K^{(i)}$ to the next $K^{(i+1)}$ with the help of the gradient method we are first assigned by initial values $K_1^0, K_2^0, \ldots, K_r^0$ and we determine the gradient of the

function

$$F(K_1, K_2, \ldots, K_r, t)$$

at this point, for which we alternately give the same small increments to parameters $K_1^0, K_2^0, \ldots, K_r^0$.

The components of gradient will be approximately equal to

$$F(K_1^{(0)}+\Delta K, K_2^{(0)}, \ldots, K_r^{(0)}) - F(K_1^{(0)}, K_2^{(0)}, \ldots, K_r^{(0)}) \approx$$
$$\approx \frac{\partial F(K_1^{(0)}, \ldots, K_r^{(0)})}{\partial K_1^{(0)}} \Delta K;$$
$$F(K_1^{(0)}, K_2^{(0)}+\Delta K, K_3^{(0)}, \ldots, K_r^{(0)}) - F(K_1^{(0)}, K_2^{(0)}, \ldots, K_r^{(0)}) \approx$$
$$\approx \frac{\partial F(K_1^0, \ldots, K_2^0)}{\partial K_2^0} \Delta K;$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$F(K_1^{(0)}, K_2^{(0)}, \ldots, K_r^{(0)}+\Delta K) - F(K_1^{(0)}, K_2^{(0)}, \ldots, K_r^{(0)}) \approx$$
$$\approx \frac{\partial F(K_1^{(0)}, \ldots, K_r^{(0)})}{\partial K_r^0} \Delta K \quad (\Delta K > 0).$$

After determining the value of gradient at the particular point we make the step in the direction, opposite to the direction of gradient, i.e., in the direction of the quickest decrease $F(K_1, K_2, \ldots, K_r)$. Next point is found by the formulas

$$\left.\begin{aligned}
K_1^{(1)} &= K_1^{(0)} - \varepsilon \frac{\partial F(K_1^0, K_2^0, \ldots, K_r^0)}{\partial K_1^{(0)}};\\
K_2^{(1)} &= K_2^{(0)} - \varepsilon \frac{\partial F(K_1^{(0)}, K_2^{(0)}, \ldots, K_r^{(0)})}{\partial K_2^{(0)}};\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
K_r^{(1)} &= K_r^{(0)} - \varepsilon \frac{\partial F(K_1^{(0)}, K_2^{(0)}, \ldots, K_r^{(0)})}{\partial K_r^{(0)}}\\
&(\varepsilon > 0).
\end{aligned}\right\} \quad \text{(VIII.28)}$$

Page 223.

At this point we compute $F(K_1^{(1)}, K_2^{(1)}, \ldots, K_r^{(1)})$ and compose the equality

$$F(K_1^{(1)}, K_2^{(1)}, \ldots, K_r^{(1)}) - F(K_1^{(0)}, K_2^{(0)}, \ldots, K_r^{(0)}). \quad \text{(VIII.29)}$$

If this equality has negative value, then we construct gradient anew. Otherwise we realize a fragmentation of step/pitch $\varepsilon/2$. At new point again we compute $F(K_1, K_2, \ldots, K_r)$:

$$\left.\begin{array}{l} K_1^{(1)} = K_1^{(0)} - \dfrac{\varepsilon}{2}\cdot\dfrac{\partial F(K_1^{(0)}, \ldots, K_2^{0})}{\partial K_1^{(0)}}; \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ K_r^{(1)} = K_r^{(0)} - \dfrac{\varepsilon}{2}\cdot\dfrac{dF(K_1^{(0)}, \ldots, K_r^{(0)})}{\partial K_r^{(0)}} \end{array}\right\} \quad \text{(VIII.30)}$$

and we investigate differences (VIII.29). We repeat these computations until difference (VIII.29) becomes negative. Analogously we find the sequence of parameters $K_1^{(i)}, K_2^{(i)}, \ldots, K_r^{(i)}$.

Each subsequent approximation/approach is found by the recurrent formula

$$K^{(i)} = K^{(i-1)} - \varepsilon\frac{\partial F(K_1^{(i-1)}, \ldots, K_r^{(i-1)})}{\partial K}; \quad \text{(VIII.31)}$$

in this task calculation it is finished, when error Y ("gross error") becomes less than the assigned magnitude.

With a considerable quantity of optimizable parameters this methodology leads to the high expenditure of machine time, since in proportion to approximation/approach to extremum F partial derivatives vanish and a number of steps/pitches unlimitedly increases.

From general method examined above of the search for the optimum parameters of the control system escapes/ensues the following computational algorithm of gradient method, which ensures convergence to the minimum (maximum) independent of the selection of the initial approximation/approach:

$$K_j^{(l+1)} = K_j^{(l)} - (r+1)\,\varepsilon^{(l)} \frac{\Delta F_{jl}}{\sqrt{\sum_{j=1}^{r} (\Delta F_{jl})^2}}, \qquad \text{(VIII.32)}$$

$$j = 1, 2, \ldots, r; \quad l = 0, 1, 2, \ldots,$$

where

$$\Delta F_{1l} = F(t, K_1^{(l)} + \Delta K^{(l)}, K_2^{(l)}, \ldots, K_r^{(l)}) - F(t, K_1^{(l)}, K_2^{(l)}, \ldots, K_r^{(l)});$$

$$\Delta F_{2i} = F\left(t,\ K_1^{(i)},\ K_2^{(i)} + \Delta K^{(i)},\ \ldots,\ K_r^{(i)}\right) - F\left(t,\ K_1^{(i)},\ K_2^{(i)},\ \ldots,\ K_r^{(i)}\right);$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Delta F_{ri} = F\left(t,\ K_1^{(i)},\ K_2^{(i)},\ \ldots,\ K_r^{(i)} + \Delta K^{(i)}\right) - F\left(t,\ K_1^{(i)},\ K_2^{(i)},\ \ldots,\ K_r^{(i)}\right);$$

$$\varepsilon^{(i)} = \begin{cases} \varepsilon^{(i-1)}, & \text{если}^{(1)}\ F_{0i} < F_{0(i-1)}; \\ \dfrac{\varepsilon^{(i-1)}}{2}, & \text{если}^{(1)}\ F_{0i} \geqslant F_{0(i-1)}; \end{cases} \qquad \text{(VIII.33)}$$

$$\Delta F_{1i} = F_{1i} - F_{0i};$$
$$\Delta F_{2i} = F_{2i} - F_{0i};$$
$$\cdots\cdots\cdots$$
$$\Delta F_{ri} = F_{ri} - F_{0i}.$$

Key: (1). if.

Page 224.

The space of computations with a large number of optimizable parameters can be substantially abbreviated/reduced, if the optimization of the parameters is realized not immediately from all parameters, but after their laying out to the groups. For example, in the first group it is possible to include/connect ($K_1$, $K_2$, $K_3$), in the second group ($K_4$, $K_5$, $K_6$), etc. After fixing, for example, the parameters in all groups, except the first, make optimization ($K_1$, $K_2$, $K_3$), then the obtained parameters ($K^*_1$, $K^*_2$, $K^*_3$) and the parameters of all other groups are fixed/recorded, besides the second, the parameters of the second group, etc optimize. The obtained values of the parameters are initial for the carrying out of

the simultaneous optimization of all parameters on the "maximum of probability". The end/lead of the calculation is determined by the conditions

$$\varepsilon^{(l)} < \delta_1; \qquad |F_{0l} - F_{0(l-1)}| < \delta_2, \qquad \text{(VIII.34)}$$

where $\delta_1$, $\delta_2$ - some sufficiently small numbers.

Formula (VIII.32) in general form is written/recorded as follows:

$$K_j^{(l+1)} = K_j^{(l)} - (r+1)\,\varepsilon^{(l)} \frac{\dfrac{\partial F}{\partial K_j^{(l)}}}{\sqrt{\sum\limits_{j=1}^{r}\left(\dfrac{\partial F}{\partial K_j^{(l)}}\right)^2}}. \qquad \text{(VIII.35)}$$

For the symmetrical function F the step/pitch $\varepsilon$ can be determined according to the formula

$$\varepsilon^{(l)} = \delta^{*}\sqrt{r}\,K_0^{l}, \qquad \text{(VIII.36)}$$

where δ - preset error in the computation of the parameters;

$K_0^{(l)}$— the initial value of the parameters;

r - number of parameters.

Page 225.

Use/application of formulas (VIII.32) or (VIII.35) in comparison

with formula (VIII.31) gives the considerable decrease of the space of computations.

An increase in the accuracy can be achieved/reached due to the variable step/pitch, when

$$\varepsilon^{(l)} = \begin{cases} \varepsilon^{(l-1)}, & \text{если } F_{0l} < F_{0(l-1)}; \\ \dfrac{\varepsilon^{(l-1)}}{k}, & \text{если } F_{0l} \geqslant F_{0(l-1)}, \end{cases} \qquad \text{(VIII.37)}$$

Key: (1). if.

where k - coefficient of the fragmentation of step/pitch.

However, the absence in this method of the necessary information about the form of the function F upon transfer from one step/pitch toward another impedes the determination of optimum values $\varepsilon^{(i)}$ and k. Therefore during the optimization of complex systems the space of computations is nevertheless large.

The considerable decrease of the space of computations is obtained during the use of the following formula, which gives information about the form of the function F:

$$K_j^{(l+1)} = K_j^{(l)} - (r+1)\gamma^{(l)}\varepsilon^{(l)} \frac{\left|\dfrac{\partial F}{\partial K_j^l}\right|}{\sqrt{\sum\limits_{j=1}^{r}\left(\dfrac{\partial F}{\partial K_j^l}\right)^2}} \operatorname{sign} \frac{\partial F}{\partial K_j^l}, \qquad \text{(VIII.38)}$$

where

$$\gamma^{(l)} = \frac{K_j^{(l+1)}}{K_j^{(l)}},$$

or

$$K_l^{i+1} = K_l^{(i)} - (r+1)\gamma^{(i)}\varepsilon^{(i)} \frac{|\Delta F_{li}|}{\sqrt{\sum_{l=1}^{r} (\Delta F_{li})^2}} \operatorname{sign} \Delta F_{li}. \qquad \text{(VIII.39)}$$

The use/application of this formula reduces the space of computations in comparison with the computations according to formula (VIII.31) approximately/exemplarily 40 times.

Substantially the decrease of the space of computations it is possible to attain also by the use/application of the method of steepest descent, which it is expedient to use, when the function of criterion F has "flat" relief.

The method of steepest descent is realized during the following selection of the sequence of points $K^{(i)}$:

$$K^{(1)} = K^{(0)} - \varepsilon H,\ K^{(2)} = K^{(0)} - 2\varepsilon H. \qquad \text{(VIII.40)}$$

The computations cease, when fulfilling of the inequalities

$$F_0(K^{(r)}) \geqslant F_0(K^{(r-1)}),\ F_0(K^{(r-1)}) < F_0(K^{(r-2)}).$$

is provided.

Page 226.

Now the value $K^{(r-1)}$ undertakes initial, and the described sequence of computations further is repeated. Here at each step the new direction of gradient also is determined and it is implemented r+1 integrations of reference system.

The combination of gradient method with the method of steepest descent can give the specific gain in the space of computations.

Let us give the short description of the algorithm of the solution of problem based on the example of the search for three parameters.

The system of the differential equations of the n order of the type of system (VIII.6), which depends on parameters $K_1$, $K_2$, $K_3$, is given. These parameters must be selected so that the function $M[\Delta Y^2]$ would take minimum value. The minimization of function $M[\Delta Y^2]$ is realized according to the method of gradient, for which the reference system of differential equations is solved by the method of Runge -

Kutta in the interval [0, T], and with the different values of the parameters are calculated from the method of statistical nodes (Chapter V) the following functions:

$$
\begin{aligned}
&M\left[\Delta Y^2 (K_1^{(0)},\ K_2^{(0)},\ K_3^{(0)})\right] = f_0;\\
&M\left[\Delta Y^2 (K_1^{(0)} + \varepsilon,\ K_2^{(0)},\ K_3^{(0)})\right] = f_1;\\
&M\left[\Delta Y^2 (K_1^{(0)},\ K_2^{(0)} + \varepsilon,\ K_3^{(0)})\right] = f_2;\\
&M\left[\Delta Y^2 (K_1^{(0)},\ K_2^{(0)},\ K_3^{(0)} + \varepsilon)\right] = f_3.
\end{aligned}
$$

Are calculated differences $(f_1-f_0)$ $(f_2-f_0)$ $(f_3-f_0)$, the necessary for the approximate computation directions of gradient.

The values of the parameters at the next step/pitch are determined from the formulas

$$
\begin{aligned}
&K_1^{(1)} = K_1^{(0)} - \varepsilon \frac{(f_1 - f_0)}{\sqrt{(f_1 - f_0)^2 + (f_2 - f_0)^2 + (f_3 - f_0)^2}};\\
&K_2^{(1)} = K_2^{(0)} - \varepsilon \frac{(f_2 - f_0)}{\sqrt{(f_1 - f_0)^2 + (f_2 - f_0)^2 + (f_3 - f_0)^2}};\\
&K_3^{(1)} = K_3^{(0)} - \varepsilon \frac{(f_3 - f_0)}{\sqrt{(f_1 - f_0)^2 + (f_2 - f_0)^2 + (f_3 - f_0)^2}}.
\end{aligned}
$$

We carry out computations as long as $|f_0^{(i-1)} - f_0^{(i)}| < \alpha$ and $\epsilon \leq \beta$. Those values of parameters $K_1$, $K_2$, $K_3$, in which are satisfied these conditions, and will be unknown.

A block diagram of calculations for $\alpha = 0{,}01$ and $\beta = 0{,}01$ is presented in Figure VIII.I.

Algorithms examined above of the method of gradient do not

possess universality for all types of the surfaces of the depicted functions of many variables due to the locality of the search for extremum.

Page 227.

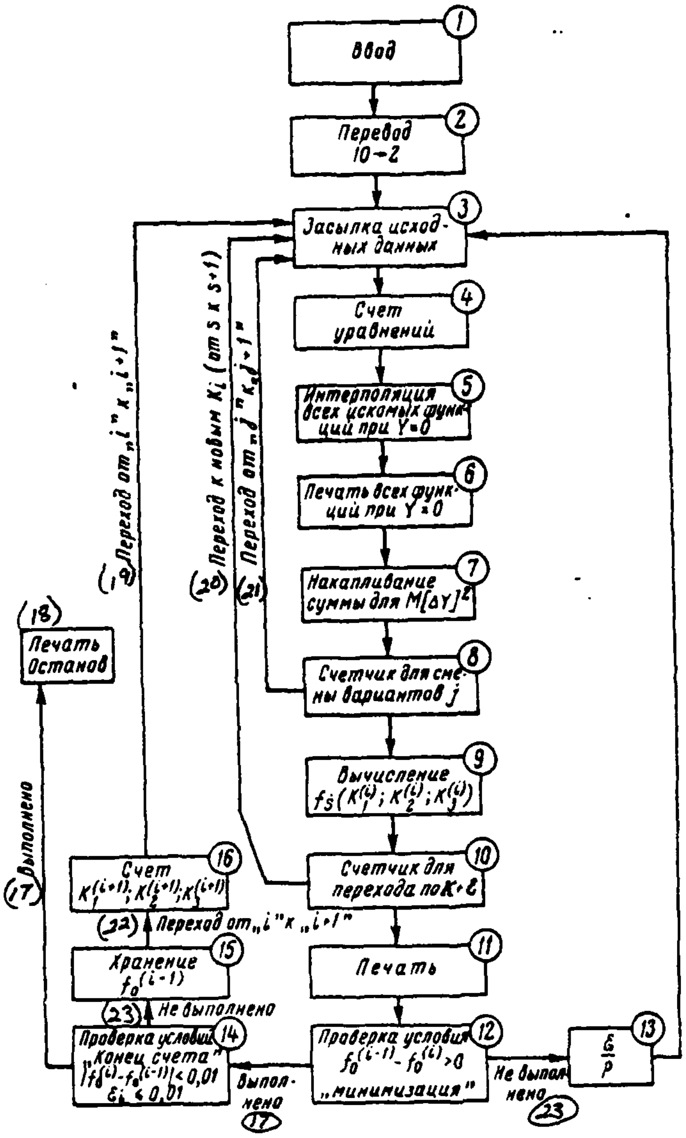



Fig. VIII.1. Block diagram of the algorithm of the optimization of system from three parameters.

Key: (1). Input/introduction. (2). Translation/conversion. (3). Dispatching of initial data. (4). Calculation of equations. 5). Interpolation of all of unknown of function with Y=0. (6). Printing all functions with Y=0. (7). Accumulation of sum for $M[\Delta Y]^2$. (8). Counter for exchanging versions j. (9). Computation. (10). Counter for transition/junction on $K+\epsilon$. (11). Printing. (12). Checking condition ... "minimization". (13). ($\epsilon$/P). (14). Checking conditions "end/lead of calculation". (15). Storage. (16). Calculation. (17). It is carried out. (18). Printing stop. (19). Transition/junction from "i" k "i+1". (20). Transition/junction to new ones ... (from s to s+1). (21). Transition/junction from "j" k "j+1". (22). Transition/junction from "i" k "i+1". (23). It is not carried out.

Page 228.

Through the method of gradient one relative extremum of function F of several possible ones is located only, i.e., there is a part of the extrema, with which the inequality

$$|E| \leqslant C_1, \quad \text{(VIII.41)}$$

is satisfied but for others the inequality

$$|E| > C_1. \quad \text{(VIII.42)}$$

occurs.

For the solution of the problem of the synthesis of system of

equations for the maximum of the probability of the nonappearance of error (gross error) E of the preset limits (0 and $C_1$ or $C_1$ and $C_2$) it suffices to find one of the existing extrema, in which is satisfied inequality (VIII.41) and corresponding to it vector K.

If the first obtained extremum does not satisfy inequality (VIII.41), then search must be begun from another point (i.e. with other initial values of parameters $K_1$, $K_2$, $K_3$), separated from the first in order to obtain extremum F, which satisfies condition (VIII.41). Theoretically it is possible to visualize the case, when the extremum, which satisfies inequality (VIII.41), under the selected law of control is not located. This it indicates that the selected law of control is not optimum and it must be reexamined.

Besides the modified algorithms of gradient method (VIII.35) examined and (VIII.40), the leading to the decrease of space computations in comparison with the generally accepted algorithm of gradient method in several dozen times, theoretically gives increase in the efficiency/cost-effectiveness of gradient method, for example, the method of improved strategy.

If the initial value of vector is equal $K^0$, and the subsequent approximation/approach is determined from formula (VIII.38), then it is possible to expand function F in Taylor series in the vicinity of

point K°, after preserving three members:

$$F(K^{(0)}-\varepsilon H)=F(K^{(0)})+\frac{\partial F(K^{(0)})}{\partial\varepsilon}\varepsilon+\frac{1}{2}\frac{\partial^2 F(K^{(0)})}{\partial\varepsilon^2}\varepsilon^2+0(\varepsilon^3). \quad \text{(VIII.43)}$$

Derivatives in the right side can be expressed as follows:

$$\frac{\partial F}{\partial\varepsilon}=\sum_{i=1}^{r}\frac{\partial F}{\partial K_i}H_i=\sum_{i=1}^{r}f_iH_i;$$

$$\frac{\partial^2 F}{\partial\varepsilon^2}=\sum_{i,j=1}^{r}\frac{\partial^2 F}{\partial K_i\partial K_j}H_iH_j=\sum_{i,j=1}^{r}f_{i,j}^{(1)}H_iH_j.$$

Task is reduced to the determination of direction H, in which the difference

$$F(K^{(0)})-F(K^{(0)}-\varepsilon H)$$

reaches maximum value.

Page 229.

As is evident, here is considered quadratic term of expansion (VIII.43), who makes it possible to improve strategy of the selection of optimum direction in comparison with the ordinary gradient method, in which is considered only second term of expansion (VIII.43).

According to Newton's method it is possible to approximately

determine the value of the step/pitch:

$$\varepsilon = \frac{\sum_{i=1}^{r} f_i H_i}{\sum_{i,j=1}^{r} f_{i,j}^{(1)} H_i H_j}.$$

After substitution and solution of problem to the maximum via the equating of particular derivatives of $[F(K^{(0)})-F(K^{(0)}-\varepsilon H)]$ on H zero will be obtained linear algebraic system (r-1) of equations with r by unknowns relatively $H_j$, whose solution determines vector H:

$$\sum_{j=1}^{r} (f_i f_{i+1,j}^{(1)} - f_{i+1} f_{i,j}^{(1)}) H_j = 0, \quad i = 1, 2, \ldots, r-1.$$

Here a number of steps/pitches and the space of computations depend on the successful selection of the initial point $K^0$ and the form of the function F.

The use of methods of the global search for the extremum of function F is of interest. Besides the method of the random search, are a net point method, "ravine" method [21], which is the combination of the random search of gradient method, method of sections, based on the determination of the form of modular surface F and assuming the "smoothness" of this relief.

Up to now still there are no waste algorithms of these methods for solving stated problem. Attempts at the use/application of these methods for the solution of the problem of the optimization of the multidimensional systems, described by the equations of high order in the presence of random interactions, lead to the volume of the computational works of the same order as as the method for statistical testing.

3. Limited synthesis of the optimum nonlinear control system on the criterion of the "maximum of probability".

The task of the limited synthesis of optimum nonlinear system of control, is of interest, when the structure of the control system is preset, and it is necessary to select the optimum parameters, which would ensure the determination of the output coordinates of object in the preset limits during the specific time with the maximum probability.

Page 230.

In the theory of control this task is formulated as the task about the "overshoots".

Below there is stated the methodology of identification of parameters on the criterion of the "maximum of the probability" of satisfaction to the limitations, superimposed on the output coordinates of system, and the example of optimization of the system of the angular stabilization of the controlled object in the digital computer is given.

The motion of the controlled object together with the control system in general form is described by the system of the differential equations

$$\frac{dY_i}{dt} = F_i(t, Y_1, Y_2, \ldots, Y_n, V_1, V_2, \ldots, V_m, K_1, K_2, \ldots, K_r), \tag{VIII.44}$$

where $Y_i$ - coordinate of the motion of object;

$K_1, K_2, \ldots, K_r$ - parameters of the system of control, which must be selected;

$V_1, V_2, \ldots, V_m$ - random variables, whose probabilistic characteristics are preset.

Limitations for the motion of apparatus they are determined by

the system of the inequalities

$$U_i(t) \leqslant Y_i(t) \leqslant W_i(t),\quad i=1, 2, \ldots, g, \qquad \text{(VIII.45)}$$

where $U_i(t)$ and $W_i(t)$ - preset functions of limitation;

g≤n - number of limited coordinates.

Limitations can be preset in the form of the one-way inequalities:

$$Y_i(t) \leqslant W_i(t),\quad i=1, 2, \ldots, g. \qquad \text{(VIII.46)}$$

The criterion of optimality - the positive functional, determined on the solution set of system (VIII.44), can be recorded in the form of the relationship/ratio

$$\Phi_i(Y_1, Y_2, \ldots, Y_n) \geqslant 0. \qquad \text{(VIII.47)}$$

Under the fixed/recorded initial conditions it is possible to examine Φ as the function of variables $V_1, V_2, \ldots, V_m$ and $K_1, K_2, \ldots, K_r$, to consider

$$\Phi_t = \Phi(V_1, V_2, \ldots, V_m, K_1, K_2, \ldots, K_r, t). \qquad \text{(VIII.48)}$$

Criterion can be probability that the values of functional lie/rest at the preset interval

$$P[0 \leqslant \Phi \leqslant C] = P(t, K_1, K_2, \ldots, K_r). \qquad \text{(VIII.49)}$$

Page 231.

The selection of the optimum parameters of system can be carried out through the maximum of the probability of satisfaction of condition (VIII.45) or through the minimum of the probability of the emergence of coordinates from region (VIII.45), i.e., in value

$$\min[1 - P].$$

Using methods of statistical nodes [109] and nonlinear programming, it is possible to determine the optimum parameters of system.

Being assigned by certain set of the statistical nodes

$$\{\mu_i\}, \quad i = 1, 2, \ldots, s, \qquad \text{(VIII.50)}$$

of the determined from the statistical characteristics disturbances/perturbations, it is necessary to obtain s of the realizations of the solutions of system (VIII.44) for the given initial values of parameters $K_j (j = 1, 2, \ldots, r)$.

Criterion $\Phi_i$ further is calculated. Then by a variation in parameters $K_i$ it is possible to obtain their values $K_i^*$, which satisfy the conditions, under which $K_i^*$ belong to the preset parametric domain $\Omega$, and the value of criterion $\Phi$ with the substitution of parameters $K_i^*$ attains extremum.

For the evaluation of the statistical characteristics of the output coordinates of the control systems let us use the method of statistical nodes, based on the approximation of the nonlinear dependences of the system being investigated in the random parameters and the interactions stochastic nodes of integration with the help of the orthogonal polynomials.

This transition/junction makes it possible to choose the approximating function, applying different measures for approximation/approach.

The polynomials of Lagrange, encompassing orthogonal polynomials of the (q+1)th order and constructed on Chebyshev's nodes, make it possible to obtain sufficiently simple expressions for determining the moments/torques of the output coordinates of the control system.

According to characteristic function $X_i$ [see formula (VIII.3)] it is possible to obtain expression for determining the probability

of the determination of coordinates in preset region (VIII.45). The process of computations can pass as follows.

Being assigned by the tabular values of the nodes of Chebyshev and numbers of Christoffel, we integrate the system (VIII.49) to the required moment of time. According to the obtained results we further determine the value of the statistical criterion, which is certain function from the obtained evaluations/estimates of system. The fact that the determination of the value of criterion easily is realized on TsVM, is the advantage of this methodology. It suffices to send each particular realization into one summing cell before obtaining of the full/total/complete value of criterion.

If controlling or exciting interactions are preset in the form of random functions, then it is necessary to switch over from the random functions to random variables by one of the known methods.

Page 232.

Applying the combined method of steepest descent and gradient, the search for the optimum parameters of system is realized by methods of nonlinear programming.

Let us give the necessary relationships/ratios for solving

stated problem.

With the help of the computers it is possible to find the set of the solutions of system (VIII.44) in Chebyshev's nodes

$$\{Y_i(t, \mu_{1\kappa_1}, \mu_{2\kappa_2}, \ldots, \mu_{m\kappa_m})\}, \quad \kappa_1, \kappa_2, \ldots, \kappa_r = 1, r, \ldots, q, \tag{VIII.51}$$

where q - order of the approximating polynomial, and to compute the characteristic function of the divergences of the output coordinates

$$X(t) = \prod_{i=1}^{q} \frac{1}{2}\left[1 - \frac{Y_i(t) - V_i(t)}{|Y_i(t) - V_i(t)|}\right] \frac{1}{2}\left[1 + \frac{Y_i(t) - U_i(t)}{|Y_i(t) - U_i(t)|}\right]. \tag{VIII.52}$$

From this equality it is evident that the characteristic function with certain assumption can take two values:

$$X(t) = \begin{cases} (1)\ 1 \text{ — при выполнении условия (VIII.45);} \\ (2)\ 0 \text{ — при нарушении хотя бы одного из неравенств} \\ \quad \text{(VIII.45).} \end{cases}$$

Key: (1). 1 - with satisfaction of condition (VIII.45). (2). 0 - during disturbance/breakdown at least of one of inequalities (VIII.45).

The probability that in the interval of time $t_1 \le t \le t_2$ the

disturbances/breakdowns of ine'.ıalities (VIII.45) will not occur, is calculated from the formula

$$P(t_1, t_2) = \lim_{\substack{N\to\infty \\ \nu\to 0}} M\left[\prod_{p=0}^{N} X(\tau_p)\right], \qquad \text{(VIII.53)}$$

where $\nu = \max|\tau_{p+1} - \tau_p|$; $\tau_p$ - moments of time characterizing by the relationship/ratio

$$\tau_0 = t_1 < \tau_1 < \tau_2 < \ldots < \tau_N = t_r.$$

With sufficiently large N

$$P(t_1, t_2) \approx M\left[\prod_{p=0}^{N} X(\tau_p)\right]. \qquad \text{(VIII.54)}$$

i.e., probability $P(t_1, t_2)$ is approximately equal to the mathematical expectation of the function

$$\Phi(t_1, t_2) - \prod_{p=0}^{N} X(\tau_p). \qquad \text{(VIII.55)}$$

Page 233.

For determining the mathematical expectation $M[\Phi(t_1, t_2)]$ sufficient, using relationships/ratios (VIII.51), to find the values of function X(t) according to formula (VIII.52)

$$\{X(\tau_p, \mu_{1\kappa_1}, \mu_{2\kappa_2}, \ldots, \mu_{r\kappa_r})\}, \qquad \text{(VIII.56)}$$
$$s = m + n; \quad \kappa_1, \kappa_2, \ldots, \kappa_r = 0, 1, 2, \ldots, q,$$

from which is calculated the set

$$\{\Phi(t, \mu_{1\kappa_1}, \mu_{2\kappa_2}, \ldots, \mu_{r\kappa_r})\}, \qquad \text{(VIII.57)}$$
$$s = m + n; \quad \kappa_1, \kappa_2, \ldots, \kappa_r = 0, 1, 2, \ldots, q.$$

In Chapter V it is shown that the mathematical expectation of function with a sufficient degree of accuracy is determined from the formula

$$M[\Phi(t_1, t_2)] \approx \sum_{\kappa_1, \kappa_2, \ldots, \kappa_r} \Phi_{\kappa_1, \kappa_2, \ldots, \kappa_r} \prod_{j=1}^{r} \rho_{j\kappa_j}, \qquad \text{(VIII.58)}$$

where $\rho_{j\kappa_j}$ - Christoffel number.

The selection of the optimum parameters of the control system can be carried out. through the maximum of the probability of satisfaction of conditions (VIII.45), i.e.,

$$\max\{P(t_1, t_2)\} \approx \max\left\{\sum_{\kappa_1, \ldots, \kappa_r} \Phi_{\kappa_1, \kappa_2, \ldots, \kappa_r} \prod_{j=1}^{r} \rho_{j\kappa_j}\right\} \qquad \text{(VIII.59)}$$

or on the minimum of the probability of the emergence of coordinates

from region (VIII.45). Let us record expression (VIII.59) in general form:

$$P(t, K_1^*, K_2^*, \ldots, K_r^*) = \max\{P(t, K_1, K_2, \ldots, K_r)\}, \quad \text{(VIII.60)}$$

where $K_1^*, K_2^*, \ldots, K_r^*$ - optimum parameters.

With the help of the computers it is possible to solve equation (VIII.60).

Thus, the methodology presented makes it possible to carry out a selection of the optimum values of the parameters of the system of control of the preset structure along the maximum of the probability of satisfaction of the limiting conditions.

Example. Let us consider the system of the angular stabilization of object, which consists of sensing elements, amplifier-converter device/equipment, which includes nonlinear logical block, and actuating elements.

The optimization even of this simple system in the presence of nonlinearity and random interactions causes great difficulties. The fuller/more total/more complete account of the objective parameters of elements/cells and instruments of system and mutual effect of control channels can lead to the even greater difficulties. Let us use the methodology presented for the solution of the problem of the

optimization of system.

Page 234.

The nonlinear equations of motion of object let us record in the following form:

$$\left.\begin{aligned}
&I_{X_1}\frac{d^2X_1}{dt^2}-(I_{X_2}-I_{X_3})\frac{dX_2}{dt}\frac{dX_3}{dt}=U_1;\\
&I_{X_2}\frac{d^2X_2}{dt^2}-(I_{X_3}-I_{X_1})\frac{dX_3}{dt}\cdot\frac{dX_1}{dt}=U_2;\\
&I_{X_3}\frac{d^2X_3}{dt^2}-(I_{X_1}-I_{X_2})\frac{dX_1}{dt}\frac{dX_2}{dt}=U_3,
\end{aligned}\right\} \qquad \text{(VIII.61)}$$

where $X_1$, $X_2$, $X_3$ - the angular coordinates of object;

$U_1$, $U_2$, $U_3$ - control function.

Let us describe one of the versions of the control system along channels $X_1$, $X_2$, $X_3$ by the equations

$$\begin{aligned}
&Y_i=A_i(X_i);\\
&U_i=B_i(Y_i)+f_i, \quad i=1, 2, 3,
\end{aligned} \qquad \text{(VIII.62)}$$

where $Y_i$ - signal at the output of the sensor of angle along the i channel;

$U_i$ - control signal, formed/shaped with i-th channel with the

relay hysteresis amplifier, included by negative aperiodic feedback with the switched time constant;

$A_i$, $B_i$ - preset functions of the i channel;

$f_i$ - i random disturbance, which has the following statistical characteristics:

$$K_i(\tau)=D_i e^{-\alpha_i|\tau|}\cos\beta_i\tau;\ m_{f_i}=0.$$

For obtaining the high accuracy of the stabilization of object at the presence of the random disturbances $f_1$, $f_2$, $f_3$ it is necessary to carry out a selection of the optimum parameters of the system of the preset structure. If necessary the system can be supplemented by compensating circuit with the transfer function

$$W_K(s)=\frac{\sum_{i=1}^{n} a_i s^i}{\sum_{j=1}^{m} b_j s^j}. \qquad \text{(VIII.63)}$$

Above it was noted that the most acceptable statistical criterion is criterion of the maximum of the probability of the nonappearance of the phase coordinates of object and energy consumption G beyond the limits of the preset borders, i.e.

$$\max P\{|X|<|C_1|,\ |\dot{X}|<|C_2|,\ |G|<C_3\}, \qquad \text{(VIII.64)}$$

where $C_1$ - requirements vector of the angular coordinates of object;

$C_2$ - vector of limitation of the angular velocities of object;

$C_3$ - limitation of energy G of the control system, spent on the control of object.

It must be noted that the limitations can be some functions of time.

Methodology presented above of the statistical synthesis of stabilization system in connection with the object in question consists of the following.

Let us assume that random functions $f_1$, $f_2$, $f_3$ can be with the help of the nonlinear transformation represented in the form of the functions of random variables and time

$$f_i(t) = \sigma_i (\sin V_{1i} t + V_{2i} \cos V_{1i} t), \quad i = 1, 2, 3 \qquad \text{(VIII.65)}$$

where

$$\sigma_i = \sqrt{K_i(0)}; \; M[V_{1i}] = 0; \; M[V_{2i}^2] = 1.$$

Page 235.

Thus, the operation of three random functions is equivalent to

six random variables $V_{1i}, V_{2i}, i = 1, 2, 3,$ the density of distribution of which

$$\left.\begin{aligned} p(v_{2i}) &= \frac{1}{\sqrt{2}} e^{-\frac{v_{2i}^2}{2}}; \\ p(v_{1i}) &= \frac{\alpha_i}{\pi} \cdot \frac{v_{1i}^2+\alpha_i^2+\beta_i^2}{(v_{1i}^2+\alpha_i^2+\beta_i^2)-4v_{1i}^2\beta_i^2}, \quad i=1, 2, 3. \end{aligned}\right\} \quad \text{(VIII.66)}$$

In this case the reference system of controls (VIII.61) (VIII.62) in the preliminarily selected parameters it is necessary to integrate in Chebyshev's nodes, obtained in six random variables

$$\{\mu_{1\kappa_1}, \mu_{2\kappa_2}, \mu_{3\kappa_3}, \mu_{4\kappa_4}, \mu_{5\kappa_5}, \mu_{6\kappa_6}\}, \quad \text{(VIII.67)}$$
$$\kappa_1 = 0, 1, 2, \ldots, q_1; \quad \kappa_2 = 0, 1, 2, \ldots, q_2; \ldots; \kappa_6 = 0, 1, 2, \ldots, q_6,$$

value of which they are determined by relations

$$x_{i\kappa_i} = \frac{1}{\pi}\left[\mathrm{arctg}\,\frac{\mu_{i\kappa_i}-\beta_i}{\alpha_i} + \mathrm{arctg}\,\frac{\mu_{i\kappa_i}+\beta_i}{\alpha_i}\right], \quad i = 1, 2, 3;$$
$$\mu_{j\kappa_j} = \sigma_j x_{j\kappa_j}, \quad j = 4, 5, 6. \quad \text{(VIII.68)}$$

In the process of integrating system (VIII.61) (VIII.62) there is monitored function $\Phi_i$ (VIII.55) and calculated according to it is the value of the probability

$$P\{|X| < C_1,\ |\dot{X}| < C_2,\ |G| < C_3\} = \frac{1}{(2\sqrt{\pi})^n} \sum_{i=1}^{q} \Phi_i \prod_{j=1}^{n} \rho_{jk_j}. \quad \text{(VIII.69)}$$

Further task of synthesis consists in the directed variation in the parameters (for example, by the combination of gradient method and method of steepest descent), their determining not only the value, but also the structure of system on selected statistical criterion (VIII.64).

The conducted investigations of a number of analogous systems shows that after the preliminary synthesis of system the large part of the processes under the operation of disturbances/perturbations falls outside the preset borders of $C_1$, $C_2$, $C_3$. After statistical synthesis it is possible to obtain system with a maximally possible probability of the determination of the phase coordinates of object in preset limitations.

4. Synthesis of nonlinear systems with digital controller in the control loop.

Digital computers (TsVM) can be used in the automatic control systems as the setting device for producing the program of control and as control of system in the locked outline for the stabilization

of system and production/consumption/generation control signal.

Page 236.

The use/application digital of controller (TsUM) in the control loop makes it possible to realize the complicated laws of control with the fuller/more total/more complete account of the properties of the object of control and characteristics of controlling and exciting interactions.

Engineering practice puts forth the task of developing the method of the synthesis of the optimum automatic systems, which have in the control loop of TsUM and a number of the preset linear and nonlinear elements/cells, such as actuating elements, measuring elements/cells, which convert devices/equipment, etc. (Fig. VIII.2).

The synthesis of complex nonlinear systems with TsUM in the control loop in the presence of random of controlling and exciting interactions is great difficulties. At present still there are no general methods of the solution of this problem. Great successes are achieved in the development of the methods of the synthesis of linear numerical controls under the known and random influences.

The solution of the problems of synthesis can go in two

directions. In the first direction with the synthesis the algorithm of digital compensator with the preset structure and the preset elements/cells of the locked control loop is determined; the secondly - from the solution of variational problem is determined the structure of closed system as a whole, i.e., transmission matrix, and then according to the known equations of single elements/cells and object of control the algorithm of TsUM is determined.

The second direction is more complicated, and completely this problem is not yet solved.

For the solution of the problem of the statistical synthesis of complex nonlinear systems with TsUM in the control loop it is possible to propose this path.

1. For linearized preset control components and controlled object by one of known methods are determined approximate structure of linear numerical control, operation algorithm of TsUM, its fundamental characteristics.

2. According to preset nonlinear characteristics of object of control and substantially nonlinear elements/cells (actuating elements, sensing elements, which convert devices/equipment) for preliminarily selected structure of TsUM connect themselves of system of control and equations of motion.

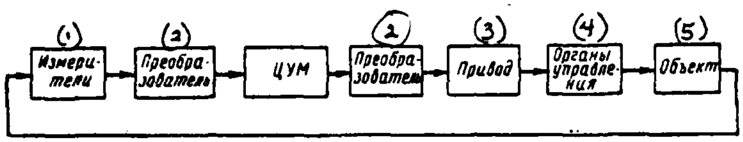



Fig. VIII.2. Block diagram of automatic system with TsUM in the control loop.

Key: (1). Meters. (2). Converter. (3). Drive. (4). Control elements. (5). Object.

Page 237.

3. According to preset statistical characteristics of random disturbances random variables, which characterize these disturbances/perturbations, are determined.

4. From precomputation and engineering considerations are assigned parameters of control components and parameters of TsUM, which must be optimized on selected criterion.

5. Selection of optimum parameters of nonlinear numerical control of involuntary (preset) structure on statistical criterion is realized by method of statistical synthesis.

The criterion of optimization can be the criterion of the

maximum of the probability of the nonappearance of the parameters of system from the preset region (for example, output coordinates or the velocities of the controlled object).

Below is stated the methodology of this statistical synthesis of nonlinear numerical controls.

Under the effect on the system of the random exciting interactions the synthesis of optimum discrete/digital nonlinear system just as continuous, can be reduced to the selection of the optimum parameters of TsUM and other elements of system on the statistical criterion of quality accepted. This method can be used for the synthesis of the algorithm of TsUM, i.e., actually for the synthesis of the law of control.

In the mathematical setting the idea of synthesis presented indicates the following.

Let the object of the control, for which it is necessary to work out discrete/digital regulator, be the dynamic component/link, which can be described in general form by equation in the vector matrix form

$$\frac{dY}{dt} \quad G[B(Y,\ t)A(U,\ t)C(F,\ t)], \qquad \text{(VIII.70)}$$

where G [B(Y, t)] - nonlinear operator with matrix/die n×m (m≤n) the controlled object;

G [A(U, t)] - nonlinear operator with matrix/die n×g (g≤n) control;

G [C(F, t)] - nonlinear operator with matrix/die n×$l$ ($l$≤n) the disturbances/perturbations, applied directly to the object;

Y - vector of the output coordinates of dimensionality n with coordinates $(Y_1, Y_2, \ldots, Y_n)$;

U - vector of the control of dimensionality r(r≤n) with coordinates $(U_1, U_2, \ldots, U_r)$;

F - vector of the exciting interactions of dimensionality $l$($l$≤n) with coordinates $(f_1, f_2, \ldots, f_l)$,

or for the simpler case

$$\frac{dY}{dt} = B(Y,\ t) + A(U,\ t) + C(F,\ t). \qquad \text{(VIII.71)}$$

Page 238.

The structural scheme of the object, described by equation (VIII.71), is depicted in Fig. VIII.3, where И is the diagonal matrix/die of size/dimension n×n, nonzero elements/cells of which are integral operators

$$И = \begin{vmatrix} \frac{1}{s} & 0 & \dots & 0 \\ 0 & \frac{1}{s} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & & \frac{1}{s} \end{vmatrix}.$$

If object is described by linear equation, then equation (VIII.71) takes the form

$$\frac{dY}{dt} = B(t)Y + A(t)U + C(t)F. \qquad \text{(VIII.72)}$$

In this case it is sometimes convenient for the research of discrete/digital system to determine the equation of the parameters of state of the object

$$Y[(\kappa+1)T] = \Phi(T)[Y[\kappa T] + a(T)U[\kappa T] + b(T)F[\kappa T]],$$
$$\kappa = 0, 1, 2, \dots, \qquad \text{(VIII.73)}$$

which is obtained from equation (VIII.72) when in the interval of time $[t_1=\kappa T,\ t_2=(\kappa+1)T]$ control pressure U and disturbance F are constant.

In equation (VIII.73) are accepted the following designations:

$$\Phi(T) = \exp\left(\int_{\kappa T}^{(\kappa+1)T} B\,dt\right) = \exp(BT);$$

$$a(T) = \int_{\kappa T}^{(\kappa+1)T} [\Phi(\kappa T - \tau)\,d\tau]\,A;$$

$$b(T) = \int_{\kappa T}^{(\kappa+1)T} [\Phi(\kappa T - \tau)\,d\tau]\,C.$$

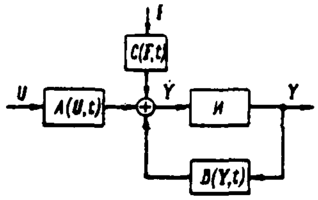



Fig. VIII.3. Block diagram of the object, described by equation (VIII.71).

Page 239.

For the one-coordinate control the object can be described by the linear rational-linear operating transfer function

$$W(s)=\frac{\sum_{i=0}^{m} a_i(t)\, s^i}{\sum_{j=0}^{n} b_j(t)\, s^j}. \qquad \text{(VIII.74)}$$

To the object it is necessary to connect discrete/digital control system. The state of the coordinates of object is determined with the help of the meters. A number of meters of coordinates $\{Y_1, Y_2, \ldots, Y_n\}$ can be h, where h≤n.

In practice not all variables can be measured directly. Therefore it is necessary to introduce the matrix/die of

transition/junction D from the parameters of state Y to the measured parameters X, due to which will be formed/shaped control U:

$$X = D(Y) + V(t), \qquad \text{(VIII.75)}$$

where $D(Y)$ — operator with the matrix/die by size/dimension m×n, preset or determined in the process of synthesis;

X - m-dimensional vector of measurements;

V(t) - the m-dimensional vector of errors of measurement.

This equality can be recorded in the form

$$X = DY + V(t), \qquad \text{(VIII.76)}$$

where D - matrix of size/dimension m×n.

The law of the formation of control U in the digital form can be assigned by the following equation:

$$U[\kappa T] = M\{K,\ X[\kappa T],\ X[(\kappa - 1)T],\ \ldots,\ X[0],\ Z[\kappa T]\}, \qquad \text{(VIII.77)}$$

where M - nonlinear operator, who determines the algorithm of the work of the control system, the part of elements/cells of which and their parameters are known;

U [$\kappa$T] - vector of the control pressures into the $\kappa$th period of

discreteness;

$Z[\kappa T]$ - vector of r-measured random function ($r \leq n$);

$X[iT]$ - vector of coordinates in the i period of discreteness ($i=\kappa, \kappa-1, \ldots, 0$);

K - some initial parameters of the systems, whose selection is necessary to carry out.

After describing the analog-digital and digital-analog converters of system respectively by the equations

$$X[\kappa T] = \varphi_1\{X(t)\}; \quad \text{(VIII.78)}$$
$$U(t) = \varphi_2\{U[\kappa T]\}, \quad \text{(VIII.79)}$$

it is possible to compose structure-matrix diagram of the control system together with the object (Fig. VIII.4).

Page 240.

Thus, the system being investigated can be described by the set of the vector-matrix continuous and discrete/digital equations or of the completely discrete/digital equations of the form

$$X[\kappa+1] = G\{\kappa, X[\kappa], \ldots, X[0], U[\kappa], K, Z(\kappa)\}, \quad \text{(VIII.80)}$$

where G - nonlinear operator with the matrices/dies of unit and discrete/digital control system.

It is obvious that from the values of parameters K the structure of the synthesized system can depend substantially.

Vector K can include any of the parameters of system, for example the period of discreteness T (which is especially importantly with the synthesis of system taking into account the operating disturbances/perturbations), or the time constants of compensating circuits, amplification factors, the measured coordinates, etc.

The task of the statistical synthesis of discrete/digital nonlinear system in accordance with the method, presented in p. 3 of Chapter VIII, can be solved as follows.

Being assigned by certain set of the vector nodes

$$\{\mu_i\} \quad (i = 1, 2, \ldots, s), \qquad \text{(VIII.81)}$$

of those determined from the statistical characteristics of the vector of disturbances/perturbations Z [$\kappa$], it is necessary to obtain s of realizations (solutions of system) (VIII.80) at the given initial values of the vector of parameters $K^0$.

The criterion

$$I=\sum_{i=1}^{s}\Phi_i\prod_{j=1}^{m}\rho_j, \qquad \text{(VIII.82)}$$

where $\Phi_i$ - characteristic function, determined by the character of criterion, further is calculated;

$\rho_j$ - weight coefficients.

Then it is possible to obtain its value of K, which satisfies the following conditions by a variation in vector $K^*$: $K^*$ belongs to the preset parametric domain Ω; the value of criterion I with the substitution of parameters $K^*$ attains extremum.

Vector $K^*$ is called optimum, and the control system, which has the vector of the parameters indicated, by optimum system.

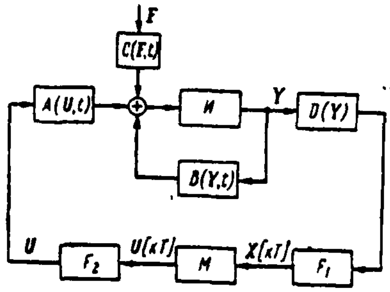



Fig. VIII.4. Structure-matrix schematic of the control system.

Page 241.

Thus, the methodology of the statistical synthesis of the nonlinear control systems, presented above, completely can be used for the synthesis of nonlinear statistically optimum systems with TsUM in the control loop.

As has already been spoken, there is a number of methods of the evaluation of the statistical characteristics of the output coordinates of the nonlinear discrete/digital and continuous control systems. However, one of promising ones is the method, based on the approximation of the nonlinear dependences of the system being investigated in the random parameters and the disturbances/perturbations stochastic nodes with the help of the orthogonal polynomials, examined in Chapters V and VI. This method makes it possible to select the approximating function, applying the

different methods of approximation/approach, for example,

$$I_1 = |\mathrm{M}[X_q - X]|; \qquad (\text{VIII}.83)$$
$$I_2 = \mathrm{M}[(X_q - X)^2]; \qquad (\text{VIII}.84)$$
$$I_3 = P[|\mathrm{M}(X_q - X)| < C], \qquad (\text{VIII}.85)$$

where X is the n-dimensional vector of the output coordinates of system;

$X_q$ - n-dimensional vector of the coordinates of system for the approximation accepted;

$$C = \text{const}.$$

After assuming that the solution of the system being investigated exists and is single for each sample of independent random quantities, it is possible to represent the unknown solution in the form of certain function of time $\kappa T$ and random variables $V_1, V_2, \ldots, V_m$,

$$X_i = X_i(\kappa T, V_1, V_2, \ldots, V_m), \qquad (\text{VIII}.86)$$
$$i = 1, 2, \ldots, n.$$

Then exact formula for computing the mathematical expectation can be recorded in the form

$$M[X_i(\kappa T, V_1, V_2, \ldots, V_m)] = \int_{\Delta} \cdots \int X_i(\kappa T, v_1, v_2, \ldots, v_m) \times \prod_{j=1}^{m} p_j(v_j)\,dv_j, \quad i = 1, 2, \ldots, n, \qquad \text{(VIII.87)}$$

where $p_j(v_j)$ - density of distribution j-ol of random variable $V_j$.

Page 242.

If we now express approximately with the required degree of accuracy function $X_i$ with the help of Lagrange's polynomials

$$X_i(\kappa T, V_1, V_2, \ldots, V_m) = \sum_{\kappa_1, \kappa_2, \ldots, \kappa_m} X_i(\kappa T, \mu_{1\kappa_1}, \ldots, \mu_{m\kappa_m}) \prod_{j=1}^{m} \frac{H_{j,q+1}(V_j)}{H'_{j,q+1}(\mu_{j\kappa_j})(V_j - \mu_{j\kappa_j})}, \qquad \text{(VIII.88)}$$

where $H_{j,q+1}(V_j)$ - orthogonal polynomials of the q+1 order, constructed by Chebyshev's nodes

$$\{\mu_{1\kappa_1}, \mu_{2\kappa_2}, \ldots, \mu_{m\kappa_m}\},$$

and to replace in formula (VIII.87) exact solution (VIII.86) with his approximation/approach (VIII.88), then, after resetting the operations of addition and integration, we will obtain

$$M[X_i(\kappa T, V_1, V_2, \ldots, V_m)] =$$

$$= \sum_{\kappa_1, \ldots, \kappa_m} X_i(\kappa T, \mu_{1\kappa_1}, \ldots, \mu_{m\kappa_m}) \int \cdots \int_{\Delta} \prod_{j=1}^{m} \frac{H_{j,q+1}(v_j) p_j(v_j)\, dv_j}{H'_{j,q+1}(\mu_{j\kappa_j})(v_j - \mu_{j\kappa_j})}, \quad \text{(VIII.89)}$$

$$i = 1, 2, \ldots, n,$$

or in the simpler form

$$M[X_i(\kappa T, V_1, V_2, \ldots, V_m)] =$$

$$= \sum_{\kappa_1, \ldots, \kappa_m} X_i(\kappa T, \mu_{1\kappa_1}, \ldots, \mu_{m\kappa_m}) \prod_{j=1}^{m} \rho_{j\kappa_j}, \quad \text{(VIII.90)}$$

where

$$\rho_{j\kappa_j} = \int_{a_j}^{b_j} \frac{H_{j,q+1}(v_j)\, p_j(v_j)}{H'_{j,q+1}(\mu_{j\kappa_j})(v_j - \mu_{j\kappa_j})}\, dv_j.$$

Formula for the approximate computation of the correlation functions of the second order can be obtained identically.

Regarding we have

$$M[X_i(\xi_1 T, V_1, V_2, \ldots, V_m)\, X_s(\xi_2 T, V_1, V_2, \ldots, V_m)] =$$

$$= \int \cdots \int_{\Delta} X_i(\xi_1 T, v_1, v_2, \ldots, v_m)\, X_s(\xi_2 T, v_1, v_2, \ldots, v_m) \prod_{j=1}^{m} p_j(v_j)\, dv_j, \quad \text{(VIII.91)}$$

$$i = 1, 2, \ldots, n.$$

Page 243.

Using a replacement in expression (VIII.91) of the product of functions $X_l$ and $X_\varkappa$ by approximation according to formula (VIII.88) and after leading the series of transformations, we will obtain

$$M\begin{pmatrix} X_l, & X_\varkappa \\ \xi_1 T, & \xi_2 T \end{pmatrix} \approx \sum_{\kappa_1, \ldots, \kappa_m} X_l(\xi_1 T, \mu_{1\kappa_1}, \ldots, \mu_{m\kappa_m}) \times X_\varkappa(\xi_2 T,$$
$$\mu_{1\kappa_1}, \ldots, \mu_{m\kappa_m}) \int_\Delta \cdots \int \prod_{j=1}^{m} \left[ \frac{H_{j,q+1}(v_j)}{H'_{j,q+1}(\mu_{j\kappa_j})(v_j - \mu_{j\kappa_j})} \right] p_j(v_j)\, dv_j. \qquad \text{(VIII.92)}$$

Then calculation formula for computing the correlation functions of the second order will take the form

$$M\begin{pmatrix} X_l, & X_\varkappa \\ \xi_1 T, & \xi_2 T \end{pmatrix} \approx \sum_{\kappa_1, \ldots, \kappa_m} X_l(\xi_1 T, \mu_{1\kappa_1}, \ldots, \mu_{m\kappa_m}) \times$$
$$\times X_\varkappa(\xi_2 T, \mu_{1\kappa_1}, \ldots, \mu_{m\kappa_m}) \prod_{j=1}^{m} \rho_{j\kappa_j}. \qquad \text{(VIII.93)}$$

Covariances of any order of the output coordinates of the control system can be determined according to the formula

$$M\left\{\begin{matrix} X_{i_1}^{\alpha_1}, X_{i_2}^{\alpha_2}, \ldots, X_{i_n}^{\alpha_n} \\ \xi_1 T, \xi_2 T, \ldots, \xi_n T \end{matrix}\right\} \approx \sum_{\kappa_1, \ldots, \kappa_m} X_{i_1}^{\alpha_1} X_{i_2}^{\alpha_2} \ldots X_{i_n}^{\alpha_n} \prod_{j=1}^{m} \rho_{j\kappa_j}, \quad (VIII.94)$$

where

$$\begin{gathered} \alpha_1 = 0, 1, 2, \ldots; \quad \alpha_2 = 0, 1, 2, \ldots; \quad \ldots; \quad \alpha_n = 0, 1, 2, \ldots; \\ \xi_1 = 0, 1, 2, \ldots; \quad \xi_2 = 0, 1, 2, \ldots; \quad \ldots; \quad \xi_n = 0, 1, 2, \ldots; \\ i_1 = 1, 2, \ldots, n; \quad i_2 = 1, 2, 3, \ldots, n; \quad \ldots, \quad i_n = 1, 2, 3, \ldots, n; \\ \kappa_1 = 0, 1, 2, \ldots, q; \quad \kappa_2 = 0, 1, 2, \ldots, q; \quad \ldots; \quad \kappa_m = 0, 1, 2, \ldots, q; \\ \xi_1 T, \xi_2 T, \ldots, \xi_n T \in [T_0, T_\kappa]. \end{gathered}$$

Page 244.

If we introduce the characteristic function

$$\Phi[\xi_i T, \xi_j T] = \prod_{\kappa=0}^{N} L(\kappa T), \qquad (VIII.95)$$

into the examination where $\xi_i T = T_0$ - beginning of process;

$\xi_j T = T_\kappa$ - end/lead of the process,

$$L(\kappa T) = \prod_{i=1}^{q} \frac{1}{2}\left[1 - \frac{X_i(\kappa T) - W_i(\kappa T)}{|X_i(\kappa T) - W_i(\kappa T)|}\right] \frac{1}{2}\left[1 + \frac{X_i(\kappa T) - U_i(\kappa T)}{|X_i(\kappa T) - U_i(\kappa T)|}\right],$$

then it is possible to obtain sufficiently simple expression for

determining the probability P of the determination of coordinates $X_i$ in the preset region:

$$U_i(\kappa T) \leqslant X_i(\kappa T) \leqslant W_i(\kappa T), \qquad \text{(VIII.96)}$$

$$P = \mathrm{M}[\Phi(\xi_i T, \xi_j T)] \approx \sum_{\kappa_1, \ldots, \kappa_r} \Phi_{\kappa_1, \ldots, \kappa_r} \prod_{j=1}^{r} \rho_{j\kappa_j}. \qquad \text{(VIII.97)}$$

Final expressions (VIII.94) and (VIII.97) permit implemention of a process of calculation as follows.

Being assigned by the tabular values of the nodes (see Chapter V), they integrate system (VIII.80) to the required moment/torque of time $\kappa$T. Further, after substituting the results of the obtained realizations into formulas (VIII.94) or (VIII.97), is determined the value of the required statistical evaluations or criterion of system. In this case one should stress that the values of Christoffel's numbers are counted previously, which considerably simplifies the procedure of the computations of probability (VIII.97).

The advantage of this methodology is the fact that the determination of the value of criterion is easy to carry out by ETsVM [99sp04 - digital computer], since it is not necessary to memorize all values of the obtained realizations during the integration of system. It suffices to send each particular realization into the summing cell before obtaining of the full/total/complete value of criterion (VIII.97).

Page 245.

Chapter IX.

# OPTIMUM SELF-TUNING CONTROL SYSTEMS.

## 1. the systems, which self-tune to the extremum of statistical criterion.

The self-tuning systems, which work with the incomplete initial information about the interactions on the fundamental control loop, are examined in this chapter. Self-adjusting in these systems is realized to the extremum of the criterion accepted: to the minimum of variance of error, to the maximum of the probability of the nonappearance of error from the preset tolerances, etc.

Let us formulate the general/common/total formulation of the problem.

We will consider that the fundamental control loop, intended for

fulfilling some functions, can be described by the nonrandom operator w (Fig. IX.1).

Let on the input of operator acts the signal X, which is the sum of useful signal G and interference Z. Let us designate the output value of fundamental control loop by Y. Different types of tasks appear with different initial data about useful signal G and about interference Z, which operate on the fundamental control loop, under different assumptions about the fundamental control loop (about operator w), with different criteria of evaluation of the quality of the work of system.

The initial data about the useful signal and about the interference are the probabilistic characteristics of these signals. With the linear operator w usually it suffices to have the first two moments/torques - mathematical expectations and the correlation functions of useful signal and interference; with the nonlinear operator w it is necessary to have laws of distribution of signals G and Z.

If the probabilistic characteristics indicated are known completely, there is no sense to use self-adjusting. But if these characteristics are known not completely, then it makes sense to attempt to use self-tuning loops for an improvement in the quality in

the work of system, i.e., for approaching the criterion of the comparison of systems to its outer limit.

The incompleteness of the information about the required probabilistic characteristics of signals G and Z can be different: a) are known all probabilistic characteristics, except the dispersion of interference; b) are known all probabilistic characteristics, except dispersion, useful signal; c) are known all probabilistic characteristics, except the dispersion of interference and dispersion of useful signal, etc.

Different cases of the incompleteness of initial information about probabilistic characteristics of useful signal and interference determine the different methods of constructing the self-tuning loops by themselves.

Fundamental control loop can be linear or nonlinear (self-tuning system as a whole always nonlinear). With the linear outline the criterion is calculated more simply, with the nonlinear - it is more complicated.

The criterion of evaluation of the quality of the work of system substantially affects self-tuning loops.

There is no common sense to develop/process in detail the structure of self-tuning system in the general case for any initial data about the probabilistic characteristics of signals, for any operator of fundamental control loop, for any criterion. To more expediently consider several typical cases. These cases encompass a large quantity of practical situations. In addition to this, at these typical cases will be explained the general principles of the construction of the self-tuning and optimum self-adjusting systems on the statistical criteria.

Before switching over to the optimum self-tuning systems, is expedient briefly to speak about the ordinary self-tuning systems.

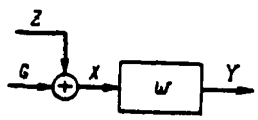



Fig. IX.1. Diagram of fundamental control loop.

Page 247.

Let us consider several cases, characterized by one or the other initial data about the probabilistic characteristics of signals, about operator w and about the criterion.

1. Law of distribution of dispersion of interference is unknown. Let to the input of fundamental control loop be fed signal X (see Fig. IX.1), which is the sum of useful signal G and interference Z. We assume that useful signal G interference Z in the calculations can be considered stationary (their probabilistic characteristics negligibly little they are changed for the control time of system).

Let the mathematical expectations of useful signal and interference be equal to zero $(m_G = m_Z = 0)$, interference and useful signal are not correlated. Consequently, the correlation function of the input signal X is equal to

$$K_X(\tau) = K_G(\tau) + K_Z(\tau). \qquad \text{(IX.1)}$$

The correlation function of useful signal is known completely, while the correlation function of interference $K_z$ takes the form

$$K_Z(\tau) = D_Z R_Z(\tau), \qquad \text{(IX.2)}$$

where $R_Z(\tau)$ - known normalized correlation function of interference;

$D_Z$ - dispersion of interference with the unknown distribution law.

However, it is known that this dispersion is changed in time slowly. For the control time in the control system it is possible to consider it constant.

Let the fundamental control loop be linear system with the preset structure and with $\kappa$ controlled parameters $x_1, \ldots, x_\kappa$.

Consequently, it is possible to consider that w is the weight function of system and that this function is known as the function of the time $\tau$ and parameters $x_1, \ldots, x_\kappa$:

$$w = w(\tau, x_1, \ldots, x_\kappa). \qquad \text{(IX.3)}$$

Let the fundamental control loop be intended for the reproduction of function H, which is the result of fulfilling the preset linear operation L above useful signal G:

$$H(t) = L\{G(t)\}. \qquad \text{(IX.4)}$$

As the criterion of system we accept the variance of error of system.

Thus, in this case there is no complete initial information about the effects on the system: the dispersion of interference $D_Z$ is unknown. In connection with this it is expedient to attempt to create the self-tuning system.

Page 248.

In order to realize self-tuning loops, it is necessary to determine the criterion of system in the process of its work. In this case it is determined so [6, 77]:

$$D_E = M[E^2] = M[(Y-H)^2] = K_Y(0) - 2K_{YH}(0) + K_H(0) = \\ = D_Y - 2D_{YH} + D_H, \quad \text{(IX.5)}$$

where $D_Y$ - dispersion of the real output value of system;

$D_H$ - dispersion of the desired output value;

$D_{YH}$ - mutual dispersion, i.e., the value of cross-correlation function $K_{YH}(\tau)$ with $\tau$=0.

Does arise the fundamental question: how to determine $D_E$ in the process of the work of system?

Let us consider separately each component/term/addend in formula (IX.5). The dispersion of real output value $D_Y$ cannot be calculated without the further working information, since it depends on unknown value - the dispersion of interference $D_Z$. Therefore it is necessary to use working information about process of Y. Should be to measure value Y and, assuming process of Y by ergodic according to the dispersion, computed dispersion $D_Y$ according to approximation formula [6, 77]

$$D_Y = \frac{1}{T_p} \int_{t}^{t+T_p} Y^2(\tau)\, d\tau, \tag{IX.6}$$

where $T_p$ - time, during which is fixed/recorded the realization of random process Y.

Value $D_{YH}$ in this case can be computed according to formula [6, 77]

$$D_{YH} = \left[ L_{t_2} \left\{ \int_0^t w(t-\tau,\ x_1,\ \ldots,\ x_\kappa)\, K_{XG}(\tau - t_2)\, d\tau \right\} \right]_{t_2 = t} = \\ = \left[ L_{t_2} \left\{ \int_0^t w(t-\tau,\ x_1,\ \ldots,\ x_\kappa)\, K_G(\underset{t_2 > T}{\tau - t_2})\, d\tau \right\} \right]_{t_2 = t}, \tag{IX.7}$$

where T - control time of fundamental control loop.

Dispersion $D_H$ is determined from formula [6, 77]

$$D_H = [L_{t_1} L_{t_2} K_G(t_1 - t_2)]_{t_1 = t_2} \tag{IX.8}$$

and it does not depend on the parameters of fundamental control loop; therefore it does not affect the construction of self-tuning loops.

Thus, using formulas (IX.6), (IX.7), (IX.8), it is possible to compute criterion (IX.5). It is the function of parameters $x_1, \ldots, x_\kappa$ of fundamental control loop:

$$D_E = D_E(x_1, \ldots, x_\kappa). \qquad \text{(IX.9)}$$

Now should be selected the method of the search for the extremum (minimum) of criterion and formed self-tuning loops.

Page 249.

Let us take, for example, gradient method [6, 49, 110] and will design the diagrams of self-tuning system (Fig. IX.2).

To the input of computer BУ are fed initial information of НИ ($K_G$, $R_Z$, $L$, ...) and the instantaneous values of controlled parameters $x_1, \ldots, x_\kappa$. Criterion $D_E$ continuously is determined at its output Synchronous detectors [6, 49, 110] or other instruments determine the instantaneous values of particular derivatives of criterion. The integrating components/links (together with the synchronous

detectors) realize the gradient method of the search for the minimum of the variance of error of system. In the diagram are not shown other the components/links, which can stand in the self-tuning loops of real system (actuating elements, converters, etc.).

The continuous process of search in accordance with the diagram in Fig. IX.2 provides the monotonic decrease of dispersion $D_{\kappa}$ under arbitrary initial conditions [5, 6, 7, 49].

In this case it was relatively easily possible to compute criterion $D_{E}$, since the correlation function of useful signal is completely known, useful signal is noncorrelated with the interference, i.e., were undertaken initial data sufficiently narrow, limited form.

Let us note one additional special feature/peculiarity of the formulation of the problem examined: in this task fundamental control loop has the preset structure, to change only the predetermined group of parameters $x_1, ..., x_n$ is possible. Obviously, with different structures of fundamental control loop and with different groups of the measured parameters different minimum values of criterion will be obtained $D_E$.

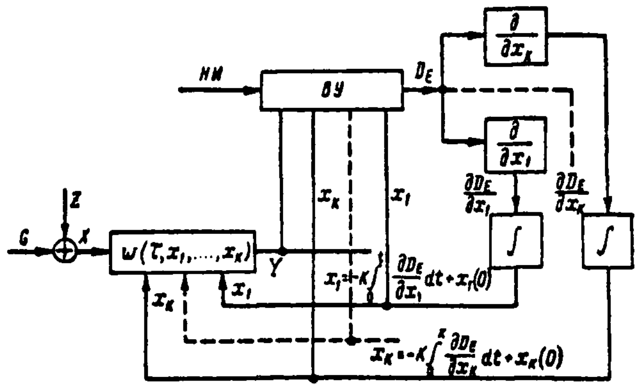



Fig. IX.2. Schematic of system with the self-adjusting according to the method of gradient.

Page 250.

The more successful are selected the structure and the controlled parameters, the less the smaller value of criterion $D_E$ the self-adjusting system can ensure.

A question about the selection of the structure of the fundamental control loop and its controlled parameters, which ensure the smallest (absolutely extreme) value of criterion, will be examined below at the examination of the optimum self-tuning systems.

The quality of the work of the self-tuning system affect also the accuracy and the high speed of the meters, executive and other

devices/equipment in the channels of self-tuning, accuracy and the high speed of computer. With the low accuracy of these devices/equipment, even in a good diagram, it can seem that the self-tuning system will be worse than the ordinary diagram. Here and throughout are not considered the further errors of these devices/equipment, i.e., we consider it their ideal. The account of the errors, introduced by the channels of self-adjusting, can be the object/subject of further experiments.

2. Law of distribution n of parameters of correlation function of interference is unknown. We consider in this case that the correlation function of interference takes the form

$$K_z = K_z(\gamma_1, \ldots, \gamma_n, \tau), \qquad \text{(IX.10)}$$

where $K_z$ - known function from the time $\tau$ and from slowly changing in the time random parameters $\gamma_1, \ldots, \gamma_n$, the law of distribution of which is unknown. Remaining initial data are the same as in the first case.

It is easy to see that in this case the self-tuning system can be designed just as in the first case (see Fig. IX.2).

Number n of parameters $\gamma_1, \ldots, \gamma_n$ can be any (and infinite). The self-tuning system can work also in the extreme case, when nothing is known about correlation function $K_z$ (i.e. if $n=\infty$). It is important

only so that the form of this function slowly would be changed in the time.

3. Law of distribution of dispersion of useful signal is unknown. The correlation function of interference is completely known in this case, And the correlation function of useful signal takes the form

$$K_G = D_G R_G(\tau), \qquad \text{(IX.11)}$$

where $R_G(\tau)$ - preset normalized correlation function of useful signal;

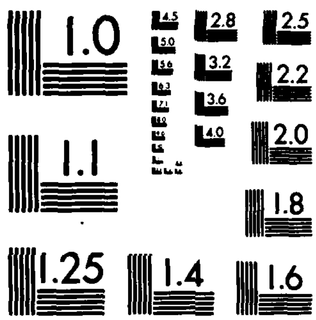
$D_G$ - dispersion of useful signal, about which it is known that it slowly is changed in the time; the law of its distribution is unknown.

Remaining initial data are the same as in the first case. In this case components/terms/addends $D_Y$ and $D_H$ in criterion (IX.5) are calculated as in the first case, according to formulas (IX.6),(IX.8).

Page 251.

The component $D_{YH}$ according to formula (IX.7) cannot be computed since function $K_G$ is completely unknown in this formula. As it follows from formula (IX.11), for total determination $K_G$ is necessary

AD-A143 297 STATIC METHODS IN THE DESIGN OF NONLINEAR AUTOMATIC 6/9
CONTROL SYSTEMS(U) FOREIGN TECHNOLOGY DIV
WRIGHT-PATTERSON AFB OH N I ANDREYEV ET AL. 27 JUN 84
UNCLASSIFIED FTD-ID(RS)T-1734-83 F/G 12/1 NL




MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

in any manner to find $D_G$. Finding $K_G$ possibly variously.

Let us consider the following path.

Dispersion $D_Y$ in the self-tuning system is determined by continuously computer according to formula (IX.6). This dispersion can be determined also according to formula [6, 5, 77]

$$D_Y = \int_0^T\int_0^T w(\tau, x_1, \ldots, x_\kappa)\, w(\lambda, x_1, \ldots, x_\kappa)\,[D_G R_G(\tau-\lambda) + K_Z(\tau-\lambda)]\,d\tau\, d\lambda = K_1(x_1, \ldots, x_\kappa)\, D_G + K_2(x_1, \ldots, x_\kappa), \quad \text{(IX.12)}$$

where

$$K_1(x_1, \ldots, x_\kappa) = \int_0^T\int_0^T w(\tau, x_1, \ldots, x_\kappa)\, w(\lambda, x_1, \ldots, x_\kappa) \times R_G(\tau-\lambda)\,d\tau\, d\lambda; \quad \text{(IX.13)}$$

$$K_2(x_1, \ldots, x_\kappa) = \int_0^T\int_0^T w(\tau, x_1, \ldots, x_\kappa)\, w(\lambda, x_1, \ldots, x_\kappa) \times K_Z(\tau-\lambda)\,d\tau\, d\lambda. \quad \text{(IX.14)}$$

From relationship/ratio (IX.12) easily is determined unknown dispersion $D_G$:

$$D_G = \frac{D_Y - K_2}{K_1}.$$

On known dispersion $D_G$, using formula (IX.7), it is possible to determine component/term/addend $D_{YH}$. Thus, diagram in Fig. IX.2 can be used also in this case. Difference from the first case lies in the fact that the computation of criterion $D_E$ is here realized by a more

complicated path.

4. Law of distribution of two parameters of correlation function of useful signal is unknown. Initial data in this case are the same as in the preceding case, with exception of the fact that the correlation function of useful signal takes another form:

$$K_G = K_G(\gamma_1, \gamma_2, \tau), \qquad \text{(IX.15)}$$

where $\gamma_1$, $\gamma_2$ - random parameters with the unknown distribution laws.

In this case before the determination of criterion it is necessary to determine the values of the parameters $\gamma_1$ and $\gamma_2$. For this it is necessary to obtain further working information, for example, to compute X dispersion $D_X$ according to the realization of process.

Page 252.

Consequently, the schematic of the self-tuning system in this case must be more complicated in comparison with the diagram, depicted in Fig. IX.2. This more compound circuit is depicted in Fig. IX.3.

Computer in this diagram continuously determines the dispersion of real output value $D_Y$ and the dispersion of input value $D_X$. The same dispersions can be calculated from the formula [5, 6, 77]

$$D_Y = \int_0^T \int_0^T w(\tau, x_1, \ldots, x_k)\, w(\lambda, x_1, \ldots, x_k) \times$$
$$+ [K_G(\gamma_1, \gamma_2, \tau - \lambda) + K_Z(\tau - \lambda)]\, d\tau\, d\lambda; \qquad \text{(IX.16)}$$
$$D_X = K_G(\gamma_1, \gamma_2, 0) + D_Z; \qquad \text{(IX.17)}$$

these two equations determine two unknown parameters $\gamma_1$, $\gamma_2$, which can be found by the cut-and-try method. In this case the parameters $\gamma_1$ and $\gamma_2$ are determined by considerably more complicated path, than parameter $D_G$ in the preceding case.

If the correlation function of useful signal contains a larger number of unknown random parameters, then for their determination it is necessary to compute additionally any probabilistic characteristics of the signals, for example, of the value of correlation functions $K_X(\tau_i)$, $K_Y(\tau_j)$ at the different values of the intervals of time $\tau_i$ and $\tau_j$. By this method to proceed is undesirable, since very long realizations of random processes are required for computing covariances of connection/communication when $\tau_i, \tau_j \neq 0$ with a sufficient accuracy.

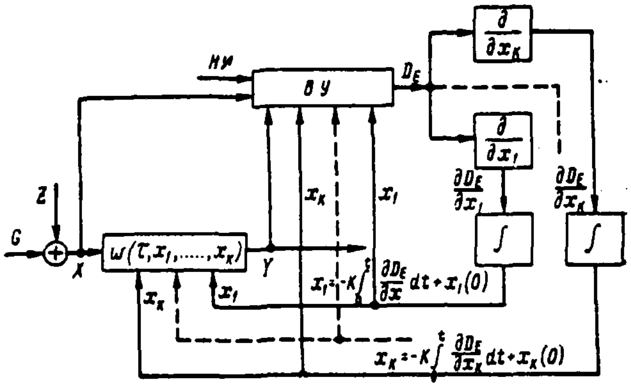



Fig. IX.3. Schematic of the self-tuning system with the determination of the parameters of useful signal.

Page 253.

Consequently, the designing of the self-tuning systems when a large number of unknown random parameters of the correlation function of useful signal is present, is qualitatively a more complex problem. The practical realization of this system is very difficult or impossible.

5. Law of distribution of dispersions of useful signal and interference is unknown. This case is the combination of the first and the third. Correlation functions $K_0$ and $K_z$ here take the following form:

$$K_0 = D_0 R_0(\tau); \qquad (IX.18)$$

$$K_z = D_z R_z(\tau), \qquad (IX.19)$$

where $D_G, D_Z$ - random parameters (dispersions) with the unknown distribution laws;

$R_G(\tau), R_Z(\tau)$ - known normalized correlation functions of useful signal and interference.

In this case during engineering of self-tuning system, it is expedient to use the diagram, depicted in Fig. IX.3. For determination $D_{YH}$ [see formula IX.7)] and $D_H$ ($D_H$ in the process of self-adjusting it does not play the significant role) it is necessary to determine $D_G$. This can be done, using values of dispersions $D_Y$ and $D_X$, the obtained in the computer, and formulas, which determine $D_Y$ and $D_X$ through $D_G$ and $D_Z$ [5, 6, 77]:

$$D_Y = K_1 D_G + K_3 D_Z; \qquad (IX.20)$$

$$D_X = D_G + D_Z, \qquad (IX.21)$$

where $K_1$ - is determined by formula (IX.13);

$$K_1 = \int_0^T \int_0^T w(\tau, x_1, \ldots, x_n) w(\lambda, x_1, \ldots, x_n) R_Z(\tau - \lambda) d\tau d\lambda. \qquad (IX.22)$$

From relationships/ratios (IX.20) and (IX.21) easily is determined dispersion $D_G$:

$$D_G = \frac{D_Y - K_3 D_Z}{K_1 - K_3}. \qquad (IX.23)$$

Now computer can compute and $D_{YH}$ according to formula (IX.7). Consequently, diagram in Fig. IX.3 makes it possible to compute criterion and its gradient.

It is easy to see that methodology presented here is applicable also to the more general case, when $K_G = K_G(\gamma_1, \tau)$ and $K_Z = K_Z(\gamma_2, \tau)$ depend on the random parameters $\gamma_1$ and $\gamma_2$ with the unknown laws of distribution.

If correlation functions $K_G, K_Z$ depend on a larger number of random parameters, then it is necessary to compute correlation functions $K_X(\tau_i), K_Y(\tau_j)$ at the different values of the intervals of time $\tau_i, \tau_j$. Deficiencies/lacks in this path were noted above.

Page 254.

6. Mathematical expectation and correlation function of interference are unknown. Criterion - probability of the nonappearance of the error of system from the preset tolerances. Initial conditions in this case in essence are the same as in the case the second. Difference lies in the fact that the mathematical expectation of useful signal $m_G \neq 0$ ($m_G$ is known), and the mathematical expectation of interference $m_Z$ is unknown. As the criterion in this case it is expedient to take the probability of the nonappearance of error from preset tolerances [1, 3]:

$$P = P(C_1 \leqslant E \leqslant C_2) = \frac{1}{\sqrt{2\pi D_E}} \int_{C_1}^{C_2} \exp\left\{-\frac{(E - m_E)^2}{2D_E}\right\} dE, \quad (IX.24)$$

where $m_E$ - mathematical expectation, it is determined from formula [6]

$$m_E=\int_0^T(m_G+m_Z)w(\tau, x_1, \ldots, x_\kappa)d\tau-L\{m_G\}=m_Y-m_H; \quad (IX.25)$$

$D_E$ - variance of error, is determined from formula (IX.5), in which instead of E, Y, H one should supply $\overset{0}{E}, \overset{0}{Y}, \overset{0}{H}$ (central random functions).

The law of the error distribution of the system is accepted normal.

For computing the criterion P it is necessary to preliminarily compute mathematical expectation $m_E$ and dispersion $D_E$ of error. Second term can be computed previously in formula (IX.25). First term $m_Y$ is expedient to compute according to the realization of the output process Y, if we assume this process ergodic. For this should be used the approximation formula

$$m_Y=\frac{1}{T_p}\int_t^{t+T_p} Y(\tau)\,d\tau. \quad (IX.26)$$

In formula (IX.5), which determines dispersion $D_E$, first addend

$$D_Y=\frac{1}{T_p}\int_t^{t+T_p}\overset{0}{Y}{}^2(\tau)d\tau=\frac{1}{T_p}\int_t^{t+T_p}[Y(\tau)-m_Y]^2\,d\tau, \quad (IX.27)$$

second term is determined from formula (IX.7).

Thus, $m_E$ and $D_E$ are determined. Computer now can compute criterion P according to formula (IX.24).

Page 255.

The schematic of the self-tuning system in this case coincides with the diagram in Fig. IX.2, only instead of criterion $D_E$ it is necessary to supply criterion P (and instead of $\frac{\partial D_E}{\partial X_i}$ to supply $\frac{\partial P}{\partial X_i}$) instead of the coefficient "-K" (K>0) one should supply "+K".

If the parameters of the correlation function of useful signal are known, then during engineering of the self-tuning system along the maximum of the probability of the nonappearance of error from the preset tolerances one should use the diagram in Fig. IX.3. With a large number of unknown parameters appear the same difficulties, as in cases of 3, 4, 5.

The fact that in them the structure of fundamental control loop is assigned arbitrarily, is the main disadvantage in self-tuning systems examined above. Therefore there is no assurance, that even with the ideal work of self-tuning loops is obtained criterion value, close to the minimum (maximum). During the successful selection of

the structure of fundamental control loop it is possible to obtain a good approximation/approach to a possible outer limit of criterion, at the unsuccessful selection of this structure (and the group of controlled parameters $x_1, ..., x_n$) the criterion of system will be distant from its extreme value.

2. Optimum systems, which self-tune to the minimum of the variance of error of system.

Different cases of the synthesis of the self-tuning systems on the statistical criteria are examined in the previous paragraph. It was explained that the computation of criterion (gradient of this criterion) is primary task in this case.

In the present paragraph will be investigated the optimum self-tuning systems, i.e., the self-tuning systems, which ensure under the given conditions the absolute extremum of the criterion accepted.

The method of computing the criterion (gradient of criterion) depends on the initial data about the conditions for the work of system. Therefore here, as in the previous paragraph, will be examined different typical cases. However, the content of this paragraph will carry more general character.

1. Law of distribution n of parameters of correlation function of interference is unknown. Let the diagram of fundamental control loop take the form, depicted in Fig. IX.1. we will consider that correlation function $K_G(\tau)$ of the stationary useful signal G(t) .is known. Correlation function $K_Z = K_Z(\gamma_1, ..., \gamma_n, \tau)$ depends on n parameters $\gamma_1, ..., \gamma_n,$ the law of distribution of which is unknown. These random parameters slowly are changed in the time (for the transit time in the fundamental control loop them it is possible to consider constants, process Z(t) it is possible to consider it virtually stationary).

Page 256.

Parameters $\gamma_1, ..., \gamma_n$ in different cases can be different characteristics of components of correlation function $K_Z$. For example, they can be to the dispersion of different components of interference. If the correlation function of interference has a form

$$K_Z = \sum_{i=1}^{M} B_i e^{-\beta_i |\tau|} \qquad \text{(IX.28)}$$

and values $B_i, \beta_i$ are unknown, then it is possible to consider that

$$\gamma_1 = B_1, \ldots, \gamma_{\frac{n}{2}} = B_M, \gamma_{\frac{n}{2}+1} = \beta_1, \ldots, \gamma_n = \beta_M.$$

Let the mathematical expectations of useful signal and

interference be equal to zero $(m_G = m_z = 0)$. Useful signal and interference are not correlated, i.e., $K_{Gz} = 0$. Consequently, the correlation function of the input signal X(t)

$$K_X(\tau) = K_G(\tau) + K_Z(\gamma_1, \ldots, \gamma_n, \tau). \qquad \text{(IX.29)}$$

Fundamental control loop is intended for the reproduction of function H($\tau$), which is the result of fulfilling certain linear operation L above useful signal G(t). Examples to functions H: H(t)=G(t); H(t)=G(t); H(t)= $G(t+T_9)$. Operation L in the specifi problem is known:

Thus,

$$H(t) = L\{G(t)\}. \qquad \text{(IX.30)}$$

In this case there is no complete initial information about the interactions, in particular, there is no complete initial information about the correlation function of interference. The formulation of the problem about the creation of the self-tuning system means makes sense.

We will study the possibility of designing of the optimum self-tuning system, which ensures the minimum of variance of error, which in this case is determined from formula [6, 60]

$$D_E = \int_0^T \int_0^T w(\tau) w(\lambda) [K_G(\tau-\lambda) + K_Z(\gamma_1, \ldots, \gamma_n \tau-\lambda)] d\tau d\lambda -$$
$$-2 \int_0^T K_{GH}(\tau) w(\tau) d\tau + D_H, \qquad \text{(IX.31)}$$

where $K_{GH}(\tau)$ - correlation function of the connection/communication between the useful signal G(t) and the desired output value H.

Taking into account expressions (IX.29) and (IX.30), it is possible to write

$$K_{GH}(\tau) = LK_G(\tau). \qquad \text{(IX.32)}$$

Page 257.

This means that correlation function $K_{GH}$ is known function from $\tau$.

If parameters $\gamma_1, \ldots, \gamma_n$ were known, then it would be possible to determine optimum weight function [6], which ensures the minimum of dispersion $D_E$. But these parameters are not known; therefore let us act as follows: let us determine optimum weight function during the replacement of unknown parameters $\gamma_1, \ldots, \gamma_n$ by some numbers $x_1, \ldots, x_n$. This means that in function $K_Z(\gamma_1, \ldots, \gamma_n, \tau)$ instead of unknown parameters $\gamma_1, \ldots, \gamma_n$ it is necessary to write numbers $x_1, \ldots, x_n$. After this it is possible to determine the optimum weight function, which is the solution of integral equation [6, 5, 60]

$$\int_0^T [K_G(\tau-\lambda)+K_Z(x_1, \ldots, x_n, \tau-\lambda)] \times$$
$$\times w(\lambda, x_1, \ldots, x_n)\, d\lambda - K_{GH}(\tau) = 0, \qquad \text{(IX.33)}$$
$$0 \leqslant \tau \leqslant T.$$

Solution $w_0$ of this integral equation depends on numbers $x_1, \ldots, x_n$:

$$w_0 = w_0(\tau, x_1, \ldots, x_n). \qquad \text{(IX.34)}$$

Being given different combinations of values $x_1, \ldots, x_n$, we will obtain optimum weight function $w_0(\tau, x_1, \ldots, x_n)$ as function $\tau$ and variable parameters $x_1, \ldots, x_n$. Obviously, the optimum weight function, which ensures the absolute minimum of dispersion $D_E$ is located in the class of these weight functions. It is easy to see that the unknown optimum weight function is obtained from general/common/total expression (IX.34) with

$$x_1 = \gamma_1, \ldots, x_n = \gamma_n. \qquad \text{(IX.35)}$$

However, parameters $\gamma_1, \ldots, \gamma_n$ are unknown. For the automatic determination of the values of parameters $\gamma_1, \ldots, \gamma_n$ or for the self-adjusting of weight function $w_0(\tau, x_1, \ldots, x_n)$ for weight function $w_0(\tau, \gamma_1, \ldots, \gamma_n)$ let us introduce self-tuning loops.

Fundamental control loop is created for this in accordance with weight function $w_0(\tau, x_1, \ldots, x_n)$. This outline must have changing

parameters (factors of amplification of components/links, time constants, etc.), which correspond to variable parameters $x_1, ..., x_n$. Weight function $w_0(\tau, x_1, ..., x_n)$ or transfer function corresponding to it determine the structure of fundamental control loop and the set/dialing of its variable parameters.

In order to synthesize self-tuning loops, it is necessary to solve the task of computing of criterion $D_K$ and its gradient. This criterion at each given moment of time can be calculated according to formula (IX.31), in which instead of the arbitrary weight function w should be supplied $w_0(\tau, x_1, ..., x_n)$, that is the solution of equation (IX.33)

$$D_E(x_1, \ldots, x_n) = \int_0^T \int_0^T w_0(\tau, x_1, \ldots, x_n)\, w_0(\lambda, x_1, \ldots, x_n) \times \times [K_G(\tau - \lambda) + K_Z(\gamma_1, \ldots, \gamma_n, \tau - \lambda)]\, d\tau\, d\lambda - -2 \int_0^T K_{GH}(\tau)\, w_0(\tau, x_1, \ldots, x_n)\, d\tau + D_H. \qquad \text{(IX.36)}$$

Page 258.

Here first term is dispersion $D_Y$ of the output value of fundamental control loop. We assume that the output value Y is stationary ergodic function (on the dispersion). In this case, feeding signal Y into the computer, on one sufficient long realization it is possible with the required accuracy to determine

unknown dispersion $D_Y$. The computation of this dispersion is realized according to formula (IX.6).

Second term in formula (IX.36) can be calculated, if we to the computer feed the values of parameters $x_1, ..., x_n$ (these values of the parameters they are known, they correspond to the real values of parameters $x_1, ..., x_n$ in the fundamental control loop).

Third component/term/addend $D_H$ does not affect the process of self-adjusting and easily it can be calculated previously, prior to the beginning of the work of system.

Thus, criterion $D_E$ can be calculated as the function of current parameters $x_1, ..., x_n$.

Further should be solved the task of determining the gradient of criterion $D_E$. For the motion to the extremum we use, for example, a gradient method. The components of gradient can be determined by the method of synchronous detection or by any other method [49, 110].

Gradient method in its ideal form in this case as follows relates rate of change in parameter $x_i$ and corresponding component of the gradient of the criterion:

$$\dot{x}_i = -K\frac{\partial D_E}{\partial x_i}, \quad i = 1, \ldots, n, \qquad \text{(IX.37)}$$

where K - certain positive number.

The schematic of the system of extreme control, which realizes the method of computing the criterion presented and the gradient method of motion to the minimum of variance of error, is depicted in Fig. IX.4. Besides the fundamental control loop, characterized by weight function $w_0(\tau, x_1, ..., x_n)$, the system encompasses other self-tuning loops. These outlines contain the elements/cells, intended for computing the criterion $D_E$, for determining the components of the gradient of criterion $\frac{\partial D_E}{\partial x_i}, i = 1, ..., n,$ for the realization of motion to the minimum of criterion.

Page 259.

Criterion $D_E$ is determined by computer and is fed into all n of the channels of self-adjusting. In the computer is fed the initial information: correlation function $K_{GH}(\tau)$; dispersion $D_H$; function $w_0(\tau, x_1, ..., x_n)$. The instantaneous values of parameters $x_1, ..., x_n$ are working information. This working information enters computer from the fundamental control loop (or from the output of the integrating components/links of the channels of self-adjusting).

The components of the gradient of criterion $\frac{\partial D_E}{\partial x_i}$ are determined continuously with the help of the synchronous detectors (or in

another manner), designated in Fig. IX.4 by rectangles $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$.

Motion to the minimum of dispersion $D_E$ is realized in accordance with formulas (IX.37) or with the formulas

$$x_i = -K\int_0^t \frac{\partial D_E}{\partial x_i} dt + x_i(0), \quad i = 1, \ldots, n. \qquad \text{(IX.38)}$$

equivalent to them.

In accordance with signals (IX.38), obtained from the output of the integrating devices/equipment, the actuating elements change parameters $x_1, \ldots, x_n$ of fundamental control loop.

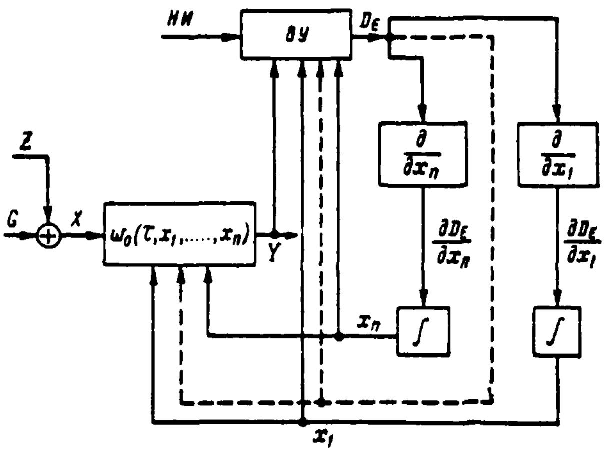



Fig. IX.4. Optimum system, which self-tunes to the minimum of variance of error.

Page 260.

The values of parameters $x_1, \ldots, x_n$, introduced into the computer, simultaneously analogously are changed. The continuous process of search in accordance with the diagram in Fig. IX.4 provides the monotonic decrease of criterion $D_E$ to its minimal value. In fact, full/total/complete derivative of criterion on the time is equal to

$$\frac{dD_E}{dt} = \frac{\partial D_E}{\partial x_1}\cdot\frac{dx_1}{dt} + \ldots + \frac{\partial D_E}{\partial x_n}\cdot\frac{dx_n}{dt}. \qquad \text{(IX.39)}$$

Derivative $\frac{\partial D_E}{\partial t}$ in the assumption is close to zero. Therefore it is not taken into consideration in expression (IX.39). Substituting relationship/ratio (IX.37) into formula (IX.39), we obtain

$$\frac{\partial D_E}{dt} = -K\left[\left(\frac{\partial D_E}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial D_E}{\partial x_n}\right)^2\right]. \qquad \text{(IX.40)}$$

Hence it is apparent that the full/total/complete derivative of criterion on the time always less or is equal to zero. Consequently, criterion $D_E$ is reduced in the process of search to the minimum value.

If function has $D_E = D_E(x_1, ..., x_n)$ several minimums, then it is necessary to find out the global minimum, by using, for example, a random assignment of the initial values of parameters $x_1, ..., x_n$. However, in the practical tasks usually with the real ranges of a change in parameters $x_1, ..., x_n$ dispersion $D_E$ has one minimum.

To the case examined here (see Fig. IX.1 and formulated initial data) is reduced also that, when the interferences, applied at different points, operate on the system.

Let us compare this version of the optimum self-tuning system with version 2, examined on page 250 of this book. System with weight function $w_0(\tau, x_1, ..., x_n)$ provides in the general case the smaller value of criterion $D_E$ with the ideal self-tuning loops, than system with weight function $w(\tau, x_1, ..., x_n)$, which corresponds to version 2 (see page 250 of this book).

In this the advantage of the optimum self-tuning system. However, the self-tuning system with weight function $w(\tau, x_1, ..., x_n)$ of fundamental control loop, as a rule, is more easily realized (this system it is selected in the class of the easily realizable systems) and it can be used for different initial data (with different correlation functions $K_G$ and $K_Z$).

It should be noted that in the optimum self-tuning system a structure of fundamental control loop and a number of its controlled parameters are uniquely determined by initial data - correlation functions $K_G$, $K_Z$ and designation/purpose of system (by form of operator L).

Page 261.

2. Law of distribution m of parameters of correlation function of useful signal is unknown. Let as before the diagram of fundamental control loop take the form, depicted in Fig. IX.1. The initial data about the interactions and the designation/purpose of system the same as in p. 1 of this paragraph, with exception of the fact that the correlation function of useful signal in this version is known with an accuracy to m parameters $v_1, ..., v_m$, and the correlation function of

interference is known completely, i.e.

$$K_G = K_G(\nu_1, \ldots, \nu_m, \tau),$$

where $K_G(\nu_1, \ldots, \nu_m, \tau)$ - known function $\tau$ and parameters $\nu_1; \ldots, \nu_m$;

$\nu_1, \ldots, \nu_m$ - the random, slowly changing parameters with the unknown distribution law.

The parameters can make sense of the dispersions of single components of random process G(t), the indices of exponential curves, which approximate correlation function $K_G$, and so forth.

It is easy to see that in this case the correlation function of the input value X is equal to

$$K_X = K_G(\nu_1, \ldots, \nu_m, \tau) + K_Z(\tau), \qquad \text{(IX.41)}$$

and the correlation function of the connection between functions X and H is determined from formula [6, 77]

$$K_{XH} \quad K_{GH} \quad LK_G(\nu_1, \ldots, \nu_m, \tau). \qquad \text{(IX.42)}$$

In contrast to previous in this case correlation function $K_{XH}$ depends not only on $\tau$, but also on unknown parameters $\nu_1, \ldots, \nu_m$. This circumstance substantially complicates the process of the synthesis of the optimum self-tuning system, which will be evident from the following presentation.

Criterion $D_E$ in this case is written/recorded in accordance with formula (IX.31), also, taking into account formulas (IX.41) and (IX.42):

$$D_E = \int_0^T \int_0^T w(\tau) w(\lambda) [K_G(\nu_1, \ldots, \nu_m, \tau - \lambda) + K_Z(\tau - \lambda)] \, d\tau \, d\lambda - \\ - 2 \int_0^T L K_G(\nu_1, \ldots, \nu_m, \tau) w(\tau) \, d\tau + D_H, \qquad \text{(IX.43)}$$

where $w(\tau)$ - the weight function of fundamental control loop.

If parameters $\nu_1, \ldots, \nu_m$ were known, then it would be possible to determine the optimum weight function of system [6, 77], which ensures the minimum of dispersion $D_E$. But these parameters are unknown in stated problem. Therefore let us act as follows.

Let us determine the optimum weight function $w_0$ during the replacement in $K_G(\nu_1, \ldots, \nu_m, \tau)$ values $\nu_1, \ldots, \nu_m$ by some numbers $x_1, \ldots, x_m$.

Page 262.

After this replacement optimum weight function is defined as the solution of integral equation [5, 6, 60, 77]

$$\int_0^T [K_G(x, \ldots, x_m, \tau-\lambda)+K_Z(\tau)]\, w(\lambda, x_1, \ldots, x_m)\, d\lambda - -LK_G(x_1, \ldots, x_m, \tau)=0, \quad 0\leqslant\tau\leqslant T. \qquad \text{(IX.44)}$$

Solution $w_0$ of this integral equation depends on $x_1, \ldots, x_m$:

$$w_0=\bar{w}_0(\tau, x_1, \ldots, x_m). \qquad \text{(IX.45)}$$

Further, being assigned by different combinations of values $x_1, \ldots, x_m$, we will obtain optimum weight function $w_0(\tau, x_1, \ldots, x_m)$ as function $\tau$ and variable parameters $x_1, \ldots, x_m$. It is obvious, in the class of these weight functions $w_0(\tau, x_1, \ldots, x_m)$ is located the optimum weight function, which ensures the absolute minimum of dispersion $D_E$ at the real values of unknown parameters $\nu_1, \ldots, \nu_m$. It is easy to note that the unknown optimum weight function is obtained from general/common/total expression (IX.45) with

$$x_1=\nu_1, \ldots, x_m=\nu_m. \qquad \text{(IX.46)}$$

Now it remained to find the method of changing the parameters $x_1, ..., x_m$ in the fundamental control loop, at which is achieved exact or approximate fulfilling of relationships/ratios (IX.46). It is possible to attain this with the help of the self-tuning loops. Fundamental control loop in accordance with weight function $w_0(\tau, x_1, ..., x_m)$ first is realized. This outline is the linear system of determinate structure with m by controlled parameters $x_1, ..., x_m$.

In order to create self-tuning loops, it is necessary to solve the task of computing the criterion $D_E$. This criterion at each given moment of time is determined by formula (IV.43), in which instead of the arbitrary weight function w should be supplied function $w_0$ $(\tau, x_1, ..., x_m)$, which is the solution of equation (IX.44):

$$D_E(x_1, \ldots, x_m) = \int_0^T \int_0^T W_0(\tau, x_1, \ldots, x_m)\, w_0(\lambda, x_1, \ldots, x_m) \times$$
$$\times [K_G(\nu_1, \ldots, \nu_m, \tau-\lambda) + K_Z(\tau-\lambda)]\, d\tau\, d\lambda -$$
$$-2\int_0^T LK_G(\nu_1, \ldots, \nu_m, \tau)\, w_0(\tau, x_1, \ldots, x_m)\, d\tau + D_H. \quad \text{(IX.47)}$$

Page 263.

First term in expression (IX.47) is dispersion $D_Y$ of the real output value of fundamental control loop. Counting this value Y of virtually stationary and ergodic on the dispersion, we can with a sufficient accuracy compute its dispersion according to sufficiently long realization Y(t). The computation of dispersion $D_Y$ is realized

according to formula (IX.6).

Second term in expression (IX.47) in contrast to the previous case cannot be calculated, that as it it depends on unknown values $\nu_1, \ldots, \nu_m$. This component/term/addend cannot be determined, also, with the help of the further measurements, since it is cross-correlation function $K_{YH}(0) = D_{YH}$ between the real and that desired by output by values, and the desired output value H cannot be measured.

Third component $D_H$, as in the previous cases, the search for the minimum of criterion does not affect, since it does not depend on the parameters of fundamental control loop.

Thus, in this case criterion $D_E$ is represented in the form

$$D_E = D_Y(x_1, \ldots, x_m) - 2D_{YH}(\nu_1, \ldots, \nu_m; x_1, \ldots, x_m) + D_H. \qquad \text{(IX.48)}$$

Dispersion $D_Y$ depends also on parameters $\nu_1, \ldots, \nu_m$, but this dependence is not considered, since it in further reasonings explicitly is not used. Dependence $D_{YH}$ on these parameters significantly affects the method of computing the criterion (gradient of criterion).

During engineering of self-tuning loops it is important to have a method of obtaining the components of gradient. Therefore we will

further seek the approximate method of computing the components of the gradient of criterion, but not criterion itself. For this instead of the exact formulas, which determine the component of the gradient of the criterion

$$\frac{\partial D_E}{\partial x_j} = \frac{\partial D_Y(x_1, \ldots, x_m)}{\partial x_j} - 2\frac{\partial D_{YH}(\nu_1, \ldots, \nu_m; x_1, \ldots, x_m)}{\partial x_j},$$

$$j = 1, \ldots, m, \qquad \text{(IX.49)}$$

and which contain unknown parameters $\nu_1, \ldots, \nu_m$, we will use the approximation formulas

$$\frac{\partial D_E}{\partial x_j} \approx \varphi_j(x_1, \ldots, x_m) = \frac{\partial D_Y(x_1, \ldots, x_m)}{\partial x_j} - $$
$$- 2\left.\frac{\partial D_{YH}(\nu_1, \ldots, \nu_m; x_1, \ldots, x_m)}{\partial x_j}\right|_{\substack{\nu_1 = x_1 \\ \cdots \\ \nu_m = x_m}}, \quad j = 1, \ldots, m. \qquad \text{(IX.50)}$$

Page 264.

Derivatives $\frac{\partial D_Y}{\partial x_j}$ can be obtained, for example, with the help of the synchronous detection. The derivatives

$$\left.\frac{\partial D_{YH}(\nu_1, \ldots, \nu_m; x_1, \ldots, x_m)}{\partial x_j}\right|_{\substack{\nu_1 = x_1 \\ \cdots \\ \nu_m = x_m}}$$

can be obtained in the computer, using working information about parameters $x_1, \ldots, x_m$ and initial information about the effects on system.

Thus, instead of the partial derivatives (IX.49) of the

criterion from parameters $x_j$ in the self-tuning system it is proposed to use functions $\varphi_j$, determined by formulas (IX.50). This approximate method of computing the components of the gradient of criterion is proposed, in the first place, because in the well working self-tuning system parameters $x_j$ are close to parameters $\nu_j$, $j=1, \ldots, m$ and the replacement of parameters $\nu_j$ with parameters $x_j$ does not lead to the large divergence of functions $\varphi_j$ from derivatives $\frac{\partial D_E}{\partial x_j}$.

In the second place, as it will be shown, the proposed approximate method of computing the components of the gradient of criterion $D_E$ it provides the process of self-adjusting, which converges to the minimum of criterion $D_E$.

Motion to the minimum of dispersion $D_E$ in this case is realized according to the modified gradient method, determined by the relationship/ratio

$$\dot{x}_j = -K\varphi_j(x_1, \ldots, x_m), \qquad (IX.51)$$

or

$$x_j = -K\int_0^t \varphi_j dt + x_j(0), \qquad (IX.52)$$

$$j = 1, \ldots, m.$$

Let us show that actually/really the motion according to this method is realized to the minimum of dispersion $D_E$ and that the state

of equilibrium of system is reached when $x_j = \nu_j$.

Page 265.

In order to determine the values of the parameters, which correspond to the state of equilibrium of system, necessary to equate to zero derivatives $x_j$ [see formulas (IX.51) and (IX.50)]:

$$\varphi_j = \frac{\partial D_Y(x_1, \ldots, x_m)}{\partial x_j} - 2\frac{\partial D_{YH}(\nu_1, \ldots, \nu_m; x_1, \ldots, x_m)}{\partial x_j}\Bigg|_{\substack{\nu_1 = x_1 \\ \cdots \\ \nu_m = x_m}} = 0,$$

$$j = 1, \ldots, m. \qquad \text{(IX.53)}$$

Let us compare system of equations (IX.53) with the equations

$$\frac{\partial D_Y(x_1, \ldots, x_m)}{\partial x_j} - 2\frac{\partial D_{YH}(\nu_1, \ldots, \nu_m; x_1, \ldots, x_m)}{\partial x_j} = 0, \; j = 1, \ldots, m, \qquad \text{(IX.54)}$$

which are obtained by the equating of zero derivatives $\frac{\partial D_E}{\partial x_j}$, $j = 1, \ldots, m$ [see expression (IX.49)].

It is easy to comprehend that equations (IX.54) are satisfied at the values of parameters $x_1 = \nu_1, \ldots, x_m = \nu_m$, since these values of the parameters correspond to the minimum of variance of error $\overline{D_E}$. Hence it follows that

$$\frac{\partial D_Y(x_1, \ldots, x_m)}{\partial x_l} - 2\frac{\partial D_{YII}(\nu_1, \ldots, \nu_m; x_1, \ldots, x_m)}{\partial x_l}\Bigg|_{\substack{\nu_1 = x_1 \\ \cdots \\ \nu_m = x_m}} = 0. \quad (IX.55)$$

In this case equations (IX.54) have the following solution:

$$x_1 = \nu_1, \ldots, x_m = \nu_m.$$

Consequently, the system of extreme control, in which the motion to the minimum of criterion $D_E(x_1, \ldots, x_m)$ is realized in accordance with formula (IX.51), provides in the steady-state mode/conditions the outer limits of parameters $x_j = \nu_j$, $j = 1, \ldots, m$ and the minimum of the variance of error of fundamental control loop

$$D_{E\,\min} = D_E(\nu_1, \ldots, \nu_m).$$

The schematic of the system of extreme control, which realizes a motion to the minimum of variance of error in accordance with law (IX.51) or (IX.52), is depicted in Fig. IX.5.

In contrast to the previous case (when is absent complete initial information about the correlation function of interference and there is the complete initial information about the useful signal) here not at all values of parameters $x_1, \ldots, x_m$ is provided the monotonic decrease of criterion $D_E$ in the process of the search for the minimum of this criterion.

Page 266.

Let us explain, what limitations should be applied on the value of parameters $x_1, \ldots, x_m$, so that the diagram in Fig. IX.5 would provide the monotonic decrease of dispersion $D_E$. For this let us write expression for the full/total/complete derivative of $D_E$ on the time, taking into account that $\frac{\partial D_E}{\partial t} \approx 0$, and taking into account relationship/ratio (IX.50) and (IX.51):

$$\frac{\partial D_E}{dt} = \sum_{j=1}^{m} \frac{\partial D_E}{\partial x_j} \cdot \frac{dx_j}{dt} =$$
$$= -K \sum_{j=1}^{m} \frac{\partial D_E}{\partial x_j} \left[ \frac{\partial D_Y}{\partial x_j} - 2 \left. \frac{\partial D_{YH}(v_1, \ldots, v_m; x_1, \ldots, x_m)}{\partial x_j} \right|_{\substack{v_1 = x_1 \\ \cdots \\ v_m = x_m}} \right]. \qquad \text{(IX.56)}$$

Taking into account expression (IX.49), formula (IX.56) can be rewritten in the following form:

$$\frac{\partial D_E}{dt} = -K \sum_{j=1}^{m} \frac{\partial D_E}{\partial x_j} \left\{ \frac{\partial D_E}{\partial x_j} + 2 \left[ \frac{\partial D_{YH}(v_1, \ldots, v_m; x_1, \ldots, x_m)}{\partial x_j} - \right. \right.$$
$$\left. \left. - \frac{\partial D_{YH}(v_1, \ldots, v_m; x_1, \ldots, x_m)}{\partial x_j} \right]_{\substack{v_1 = x_1 \\ \cdots \\ v_m = x_m}} \right\}.$$

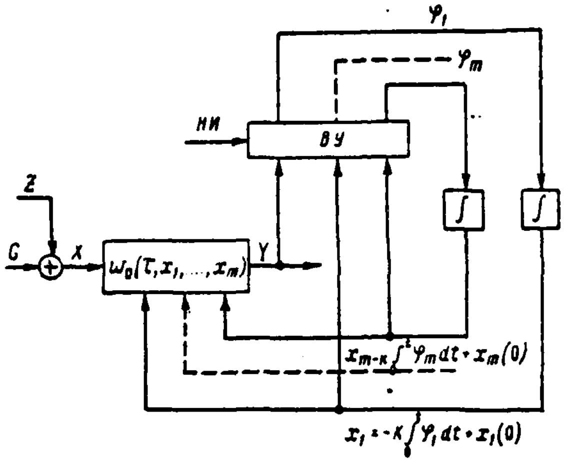



Fig. IX.5. Diagram of extreme control according to the law, described by equation (IX.51) or equation (IX.52).

Page 267.

It follows from the latter/last relationship/ratio that criterion $D_E$ monotonically decreases, if

$$\left|\frac{\partial D_E}{\partial x_l}\right| > 2\left|\frac{\partial D_{YH}(\nu_1, \ldots, \nu_m; x_1, \ldots, x_m)}{\partial x_l} - \frac{\partial D_{YH}(\nu_1, \ldots, \nu_m; x_1, \ldots, x_m)}{\partial x_l}\right|_{\substack{\nu_1 = x_1 \\ \cdots \\ \nu_m = x_m}} . \qquad \text{(IX.57)}$$

This condition is satisfied with a sufficient proximity of parameters $x_i$ to parameters

$$\nu_l, \; l = 1, \ldots, m.$$

With the initial data examined, when m>2, ordinary (not optimum)

self-tuning system cannot function without the further measurements and the computations (see Section 4 of previous paragraph). In this case in the optimum self-tuning system it is possible to approximately compute the gradient of criterion, because the outer limits of parameters $x_1, \ldots, \dot{x}_m$ coincide with the instantaneous values of unknown parameters $\nu_1, \ldots, \nu_m$.

It is important for the normal work of the optimum self-tuning system in the case in question so that the initial values of parameters $x_1, \ldots, x_m$ would be established/installed by sufficiently close to the instantaneous values of parameters $\nu_1, \ldots, \nu_m$. Sufficient proximity $x_j(0)$ to $\nu_j$, $j = 1, \ldots, m$ can be rated/estimated by relationships/ratios (IX.57).

3. Law of distribution m of parameters of correlation function of useful signal and n of parameters of correlation function of interference is unknown. In the case

$$\left.\begin{aligned} K_G &= K_G(\nu_1, \ldots, \nu_m, \tau); \\ K_Z &= K_Z(\gamma_1, \ldots, \gamma_n, \tau); \\ K_X &= K_G(\nu_1, \ldots, \nu_m, \tau) + K_Z(\gamma_1, \ldots, \gamma_n, \tau); \\ K_{XH} &= K_{GH} = LK_G(\nu_1, \ldots, \nu_m, \tau). \end{aligned}\right\} \quad \text{(IX.58)}$$

in question.

Criterion $D_E$ in this case is written/recorded according to formula (IX.31) taking into account expression (IX.58) in the following the form:

$$D_E=\int_0^T\int_0^T w(\tau)w(\lambda)\left[K_G(\nu_1,\ \dots,\ \nu_m,\ \tau-\lambda)+K_Z(\gamma_1,\ \dots,\ \gamma_n,\ \tau-\lambda)\right]\times$$
$$\times d\tau\, d\lambda-2\int_0^T LK_G(\nu_1,\ \dots,\ \nu_m,\ \tau)\,w(\tau)\,d\tau+D_H=D_Y-D_{YH}+D_H, \qquad \text{(IX.59)}$$

where $w(\tau)$ - the weigh+ function of fundamental control loop.

Page 268.

For known parameters $\nu_1, \dots, \nu_m, \gamma_1, \dots, \gamma_m$ it is possible to determine the optimum weight function of system [6, 60], which ensures the minimum of dispersion $D_E$. However, these parameters are unknown in the task in question. Let us act analogously how they entered in the previous cases: let us determine first optimum weight function $w_*$ during the replacement in function $K_G(\nu_1, \dots, \nu_m, \tau)$ of values $\nu_1, \dots, \nu_m$ by numbers $x_1, \dots, x_m$ and in function $K_Z(\gamma_1, \dots, \gamma_n, \tau)$ of values $\gamma_1, \dots, \gamma_n$ by numbers $y_1, \dots, y_n$. After this replacement the conditional optimum weight function $w_*$ is defined as solution of integral equation [5, 6, 60]

$$\int_0^T\left[K_G(x_1,\ \dots,\ x_m,\ \tau-\lambda)+K_Z(y_1,\ \dots,\ y_n,\ \tau-\lambda)\right]w\times$$
$$\times(\lambda,\ x_1,\ \dots,\ x_m,\ y_1,\ \dots,\ y_n)\,d\lambda-LK_G(x_1,\ \dots,\ x_m,\ \tau)=0,$$
$$0\leqslant\tau\leqslant T. \qquad \text{(IX.60)}$$

Solution $w_*$ of this integral equation depends on numbers $x_1, \dots, x_m, y_1, \dots, y_n$.

If we are be given different combinations of the values of numbers $x_1, \ldots, x_m, y_1, \ldots, y_n$, then we will obtain optimum weight function $w_0(x_1, \ldots, x_m, y_1, \ldots, y_n)$ as function $\tau$ and parameters $x_1, \ldots, x_m, y_1, \ldots, y_n$. It is obvious, in the class of these conditional optimum weight functions is located the optimum weight function, which ensures the absolute minimum of dispersion $D_E$ at the real value of unknown parameters $\nu_1, \ldots, \nu_m, \gamma_1, \ldots, \gamma_n$ of the probabilistic characteristics of signals G(t) and Z(t). It is easy to see that the unknown optimum weight function is obtained from the general/common/total expression of the conditional optimum weight function

$$w_0(\tau, x_1, \ldots, x_m, y_1, \ldots, y_n) \qquad (IX.61)$$

with

$$x_1 = \nu_1, \ldots, x_m = \nu_m, y_1 = \gamma_1, \ldots, y_n = \gamma_n. \qquad (IX.62)$$

It is necessary to find the method of achieving relationships/ratios (IX.62), i.e., to find the method of an automatic change of parameters $x_1, \ldots, x_m, y_1, \ldots, y_n$ also, in the fundamental control loop, at which is achieved exact or approximate fulfilling of relationships/ratios (IX.62). It is possible to attain this with the help of the self-tuning loops.

Fundamental control loop in accordance with weight function $w_0(\tau, x_1, \ldots, x_m, y_1, \ldots, y_n)$first is realized. In this outline must be n+m of controlled parameters $x_1, \ldots, x_m, y_1, \ldots, y_n$. For the creation of self-tuning

loops it is necessary to solve the task of computing of criterion $D_E$ or its gradient. Criterion at each moment of time is determined by formula (IX.59), in which instead of the arbitrary weight function w should be supplied function $w_0(\tau, x_1, ..., x_m, y_1, ..., y_n)$, which is the solution of equation (IX.60):

$$D_E = \int_0^T\int_0^T w_0(\tau, x_1, \ldots, x_m, y_1, \ldots, y_n)\, w_0(\lambda, x_1, \ldots, x_m, y_1, \ldots, y_n)\times$$
$$\times[K_G(\nu_1, \ldots, \nu_m, \tau-\lambda) + K_Z(\gamma_1, \ldots, \gamma_n, \tau-\lambda)]\, d\tau\, d\lambda -$$
$$-2\int_0^T LK_G(\nu_1, \ldots, \nu_m, \tau)\, w_0(\tau, x_1, \ldots, x_m, y_1, \ldots, y_n)\, d\tau +$$
$$+ D_H \quad D_Y - 2D_{YH} + D_H. \qquad \text{(IX.63)}$$

Page 269.

This criterion is the function of known parameters $x_1, ..., x_m, y_1, ..., y_n$ and unknown parameters $\nu_1, ..., \nu_m, \gamma_1, ..., \gamma_n$.

First term in expression (IX.63), as in the previous cases, it is calculated in formula (IX.6), where Y(t) - the realization of the real output value of fundamental control loop.

Second term cannot be computed. As in the preceding case, we will calculate approximately not criterion $D_E$, but its gradient. For this instead of the exact (unrealizable) formulas, which determine the components of the gradient of the criterion

$$\frac{\partial D_E}{\partial x_j}=\frac{\partial D_Y(x_1,\ldots,x_m,\ y_1,\ldots,y_n)}{\partial x_j}-$$
$$-2\frac{\partial D_{YH}(\nu_1,\ldots,\nu_m,\ x_1,\ldots,x_m,\ y_1,\ldots,y_n)}{\partial x_j},\ j=1,\ \ldots,\ m;\quad \text{(IX.64)}$$
$$\frac{\partial D_E}{\partial y_l}=\frac{\partial D_Y(x_1,\ldots,x_m,\ y_1,\ldots,y_n)}{\partial y_l}-$$
$$-2\frac{\partial D_{YH}(\nu_1,\ldots,\nu_m,\ x_1,\ldots,x_m,\ y_1,\ldots,y_n)}{\partial y_l},\ l=1,\ \ldots,\ n,\quad \text{(IX.65)}$$

containing unknown parameters $\nu_1, \ldots, \nu_m, \gamma_1, \ldots, \gamma_n$, we will use the approximation formulas

$$\frac{\partial D_E}{\partial x_j}=\frac{\partial D_Y(x_1,\ldots,x_m,\ y_1,\ldots,y_n)}{\partial x_j}-$$
$$-2\left.\frac{\partial D_{YH}(\nu_1,\ldots,\nu_m,\ x_1,\ldots,x_m,\ y_1,\ldots,y_n)}{\partial x_j}\right|_{\substack{\nu_1=x_1\\ \cdots\\ \nu_m=x_m}}=$$
$$=\varphi_j(x_1,\ \ldots,\ x_m,\ y_1,\ \ldots,\ y_n),\quad j=1,\ \ldots,\ m;\quad \text{(IX.66)}$$
$$\frac{\partial D_E}{\partial y_l}=\frac{\partial D_Y(x_1,\ldots,x_m,\ y_1,\ldots,y_n)}{\partial y_l}-$$
$$-2\frac{\partial D_{YH}(x_1,\ldots,x_m,\ x_1,\ldots,x_m,\ y_1,\ldots,y_n)}{\partial y_l}=$$
$$=\psi_l(x_1,\ \ldots,\ x_m,\ y_1,\ \ldots,\ y_n),\quad l=1,\ \ldots,\ n.\quad \text{(IX.67)}$$

Page 270.

Derivatives $\frac{\partial D_Y}{\partial x_j}$, $\frac{\partial D_Y}{\partial y_l}$ can be obtained, for example, with the help of synchronous detection. The derivatives

$$\left.\frac{\partial D_{YH}(\nu_1,\ldots,\nu_m,\ x_1,\ldots,x_m,\ y_1,\ldots,y_n)}{\partial x_j}\right|_{\substack{\nu_1=x_1\\ \cdots\\ \nu_m=x_m}};$$

$$\frac{\partial D_{YH}(x_1,\ldots,x_m;\ x_1,\ldots,x_m;\ y_1,\ldots,y_n)}{\partial y_l}$$

can be found by calculation in the computer, using initial information about the effects on system and working information about the parameters of fundamental control loop $x_1, \ldots, x_m, y_1, \ldots, y_n$.

Thus, instead of the partial derivatives (IX.64) and (IX.65) of the criterion from parameters $x_j$ and $y_i$ in the self-tuning system it is proposed to use functions $\varphi_j$ and $\psi_i$, determined by formulas (IX.66) and (IX.67).

The proofs of this replacement is accurate the same as in the previous case.

Motion to the minimum of dispersion $D_E$ in this case is realized according to the modified gradient method, determined by the relationships/ratios

$$\left.\begin{aligned} \dot{x}_j &= -K\varphi_j(x_1, \ldots, x_m, y_1, \ldots, y_n), \quad j = 1, \ldots, m; \\ \dot{y}_i &= -K\psi_i(x_1, \ldots, x_m, y_1, \ldots, y_n), \quad i = 1, \ldots, n. \end{aligned}\right\} \quad \text{(IX.68)}$$

In accordance with formulas (IX.63) (IX.66) (IX.67) (IX.68) the schematic of the optimum self-tuning system, depicted in Fig. IX.6, is formed/shaped. The principle of the work of system with a similar diagram is clear without the further description.

Just as in the preceding case, it is possible to show that the

equilibrium state of system is reached when $x_j = v_j$, $y_i = v_i$, the state of equilibrium of system corresponds to the minimum of dispersion $D_E$, and in the observance of conditions, analogous conditions (IX.55), motion to the minimum of dispersion $D_E$ is realized monotonically.

It was everywhere accepted until now that useful signal G and interference Z were not correlated. This assumption was accepted only for simplification in the calculations. In the latter case it was possible not to accept this assumption. In this case in principle nothing was changed, it is only necessary to know correlation function of connection/communication $K_{GZ}$ between G(t) and Z(t) as the function of some parameters $\lambda_1, \ldots, \lambda_s$:

$$K_{GZ} = K_{GZ}(\lambda_1, \ldots, \lambda_s, \tau).$$

Page 271.

All reasonings and calculations of this point/item without the fundamental changes are valid in this more general case (when $K_{GZ} \neq 0$), only more complicatedly are determined correlation functions $K_X$ and $K_{YH}$ and criterion $D_E$, the weight function $w_0$ of fundamental control loop depends on a large number of unknown parameters.

Everywhere above it was considered that mathematical expectation of useful signal and interference were equal to zero, and as the

criterion accepted the variances of error of system. Sometimes it is expedient to consider the presence not of the equal to zero mathematical expectation of useful signal or interference. In such cases it is expedient to use another criterion of the quality of systems. It is possible to take as the criterion, for example, root-mean-square error or the probability of the nonappearance of the error of system from the preset tolerances. In the following paragraph the system, which self-tunes to maximum of the probability of the nonappearance of its error from the preset tolerances, will be examined.

3. Optimum systems, which self-tune to maximum of the probability of the nonappearance of error from the preset tolerances.

Let us examine two cases, that differ one from another in terms of the degree of the incompleteness of initial information about the probabilistic characteristics, which operate on the system of signals.

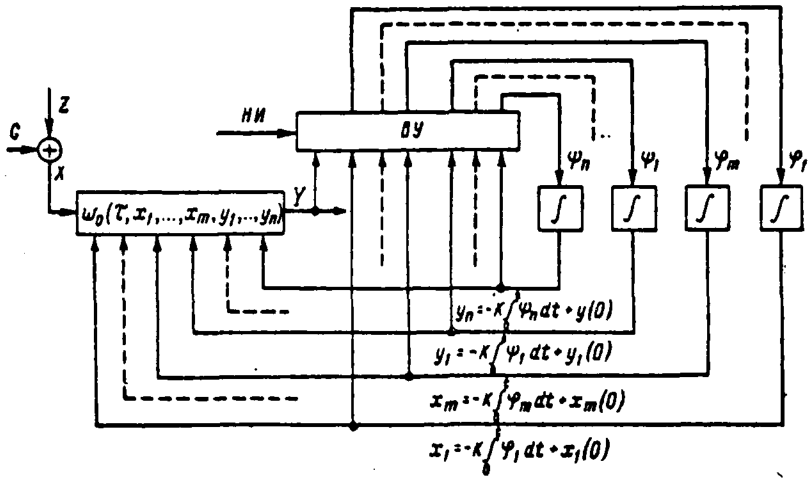



Fig. IX.6. Schematic of the optimum system, self-adjusting according to laws (IX.66)-(IX.68).

Page 272.

1. Law of distribution of mathematical expectation of interference and n of parameters of correlation function of interference is unknown. To the input of fundamental control loop (see Fig. IX.1) operates signal X, which is the sum of useful signal G and interference Z, not correlated between themselves. Mathematical expectation $m_G$ and correlation function $K_G$ of useful signal is completely known. Are unknown mathematical expectation $m_z$ and n of parameters $\gamma_1, ..., \gamma_n$ of correlation function $K_z(\gamma_1, ..., \gamma_n, \tau)$ interferences. In other respects initial data coincide with the first case of the previous paragraph.

As the criterion we accept probability P of the nonappearance of the error of system from the data of tolerances, determined under the normal law of error distribution according to formula (IX.24).

Criterion P is function from mathematical expectation $m_E$ and dispersions $D_E$ of error.

The mathematical expectation of error is determined from formula [6, 60]

$$m_E = m_Y - m_H - \int_0^T (m_G + m_Z)\,w(\tau)\,d\tau - L\{m_G\}, \quad (IX.69)$$

but dispersion $D_E$ - according to formula (IX.31).

If parameters $m_Z, \gamma_1, ..., \gamma_n$ were known, then it would be possible to determine the optimum weight function $w_0$ of fundamental control loop [6], which ensures the maximum of criterion P.

So that it would be possible to use the methodology of the determination of optimum weight function from criterion P, let us assign the concrete/specific/actual values of values $m_Z, \gamma_1, ..., \gamma_n$; instead of $m_Z$ let us take number $x_0$, instead of $\gamma_1, ..., \gamma_n$ - respectively number $x_1, ..., x_n$. Numbers $x_0, x_1, ..., x_n$ can be selected arbitrarily from the region of the possible values of values $m_Z, \gamma_1, ..., \gamma_n$. Now it is possible to determine [6] the conditional optimum weight function $w_0$ of fundamental control loop, which ensures the maximum of criterion P

when $m_z = x_0, \gamma_1 = x_1, \ldots, \gamma_n = x_n$. The weight function $w_0$ depends on $\tau$ and on numbers $x_0, x_1, \ldots, x_n$:

$$w_0 = w_0(\tau, x_0, x_1, \ldots, x_n). \qquad \text{(IX.70)}$$

Being given different combinations of values $x_0, x_1, \ldots, x_n$, we will obtain the optimum weight function $w_0$ as function $\tau$ and parameters $x_0, x_1, \ldots, x_n$. It is obvious that the optimum weight function, which ensures the absolute maximum of criterion P, is located in the class of these weight functions. The unknown optimum weight function is obtained from general/common/total expression (IX.70) at the following values of parameters $x_0, x_1, \ldots, x_n$:

$$x_0 = m_z, \; x_1 = \gamma_1, \; \ldots, \; x_n = \gamma_n.$$

Page 273.

However, parameters $m_z, \gamma_1, \ldots, \gamma_n$ are unknown. For the automatic determination of the values of parameters $m_z, \gamma_1, \ldots, \gamma_n$ or for the self-adjusting of fundamental control loop let us introduce the self-tuning loops.

In accordance with weight function $w_0(\tau, x_0, x_1, \ldots, x_n)$ is created the fundamental control loop, which is linear system with variable (adjustable) parameters $x_0, x_1, \ldots, x_n$. Then it is necessary to compute criterion P. This criterion at each current time can be calculated

according to formulas (IX.24) (IX.31) and (IX.69), where instead of the arbitrary function w one should supply $w_0(\tau, x_0, x_1, \ldots, x_n)$. The computation of variance of error $D_E$ is realized, just as this is described above in p. 1 of paragraph 2, and mathematical expectation $m_E$ - according to the formulas p. 6 of paragraph 2. For the realization of motion to the maximum of criterion P it is possible to use a gradient method. In any manner, for example with the help of the synchronous detection, should be determined the components of the gradient of the criterion

$$\frac{\partial P}{\partial x_0}, \frac{\partial P}{\partial x_1}, \ldots, \frac{\partial P}{\partial x_n}.$$

In this case the ideal form of this method is realized by the following formulas:

$$\dot{x}_i = K\frac{\partial P}{\partial x_i}, \quad i = 0, 1, \ldots, n, \qquad \text{(IX.71)}$$

where K - certain positive number, or

$$x_i = K\int_0^t \frac{\partial P}{\partial x_i}\,dt + x_i(0), \quad i = 0, 1, \ldots n. \qquad \text{(IX.72)}$$

In accordance with law (IX.71) the actuating elements of self-tuning loops change the values of parameters $x_i$ in the fundamental control loop and in the computer VU.

The schematic of the self-tuning system, which realizes the method of computing the criterion P presented and the gradient method of motion to the maximum of criterion P, is depicted in Fig. IX.7.

The continuous process of the search for the maximum of criterion P in accordance with diagram IX.7 provides the monotonic increase of criterion P under the arbitrary initial conditions. In order to be convinced of this, let us write full/total/complete derivative of criterion P on the time, taking into account that $(\partial P/\partial t)=0$:

$$\frac{dP}{dt}\sum_{i=0}^{n}\frac{dP}{\partial x_i}\cdot\frac{\partial x_i}{dt}=K\sum_{i=0}^{n}\left(\frac{\partial P}{\partial x_i}\right)^2. \qquad \text{(IX.73)}$$

Page 274.

It is evident from expression (IX.73) that the total derivative of criterion P on the time is positive, if $x_0, x_1, \ldots, x_n$ are not extreme. This means that criterion P grows monotonically in the process of search.

Let us consider the more general case of the formulation of the problem.

2. Are unknown mathematical expectation and parameters of correlation functions of useful signal and interference. In contrast to the previous case we consider that they are unknown and mathematical expectation $m_G$ and m of parameters $v_1, \ldots, v_m$ of correlation

function $K_G(\nu_1, ..., \nu_m, \tau)$ of useful signal.

Thus, in the examined case $m_G$, $m_Z$ are unknown, $K_G$, $K_Z$, $K_X$, $K_{XH}$ are determined by relationships/ratios (IX.58). Criterion P is determined from formulas (IX.24) (IX.59) (IX.69). Since parameters $m_G, \nu_1, ..., \nu_m, m_Z, \gamma_1, ..., \gamma_n$ are unknown, then the optimum weight function of system is impossible to determine (there is no complete initial information). Let us assign the concrete/specific/actual values of these parameters:

$$m_G = x_0,\ \nu_1 = x_1,\ \ldots,\ \nu_m = x_m,\ m_Z = y_0,$$
$$\gamma_1 = y_1,\ \ldots,\ \gamma_n = y_n.$$

At these values let us determine the conditional optimum weight function of fundamental control loop $w_0$ employing known procedure [6], which depends on values $x_0, x_1, ..., x_m, y_0, y_1, ..., y_n$:

$$w_0 = w_0(\tau,\ x_0,\ x_1,\ \ldots,\ x_m,\ y_0,\ y_1,\ \ldots,\ y_n). \qquad \text{(IX.74)}$$

If we are be given different combinations of the values of numbers $x_1, ..., x_m, y_0, y_1, ..., y_n$, then in this case we will obtain the optimum weight function $w_0$ as function $\tau$ and parameters $x_0, x_1, ..., x_m, y_0, y_1, ..., y_n$.

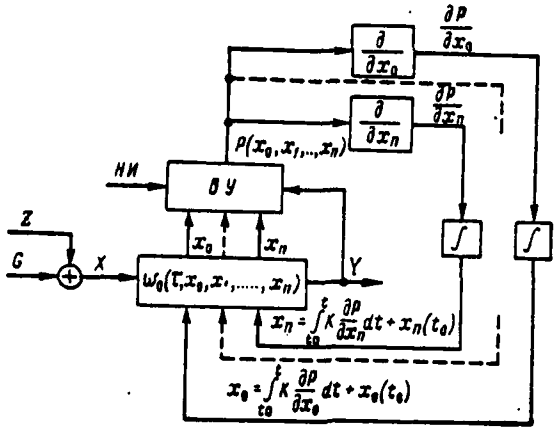


Fig. IX.7. Schematic of the self-tuning system, which realizes law (IX.72).

Page 275.

Obviously, the optimum weight function, which ensures absolute maximum of criterion P at the real values of unknown parameters $m_G, \nu_1, \ldots, \nu_m, m_Z, \gamma_1, \ldots, \gamma_n$ Is located in the class of these conditional optimum weight functions It is easy to note that the unknown optimum weight function is obtained from function (IX.74) with

$$\left.\begin{aligned} x_0 = m_G,\ x_1 = \nu_1,\ \ldots,\ x_m = \nu_m; \\ y_0 + m_Z,\ y_1 = \gamma_1,\ \ldots,\ y_n = \gamma_n. \end{aligned}\right\} \qquad \text{(IX.75)}$$

It is necessary to find the method of achieving relationships/ratios (IX.75), i.e., to find the method of an automatic change of parameters $x_i$, $y_j$ in the fundamental control loop, at which is achieved exact or approximate fulfilling of

relationships/ratios (IX.75). It is possible to attain this with the help of the self-tuning loops.

Fundamental control loop in accordance with weight function (IX.74) first is realized. In this outline must be n+m+2 controlled parameters $x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_n$. For the creation of self-tuning loops we will as in p. 3 of previous paragraph, calculate approximately the gradient of criterion.

Criterion P, determined from formulas (IX.24) (IX.59) (IX.69), is known function from mathematical expectation $m_E$ and dispersions $D_E$ of error. Mathematical expectation $m_E$ easily is determined on the realization of the output value Y of fundamental control loop taking into account formula (IX.69). Dispersion $D_E$ according to formula (IX.59) cannot be completely defined, since second term in expression (IX.59) does not depend on $w_0$. The first term $D_Y$ is defined on realization Y completely, and second term $D_{YH}$ it can be calculated as the function of parameters $x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_n$.

Consequently, criterion P can be calculated as the known function of parameters $v_1, \ldots, v_m, x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_n$:

$$P = P(v_1, \ldots, v_m, x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_n). \quad \text{(IX.76)}$$

Approximately the components of the gradient of criterion P it

is proposed to determine from the following formulas:

$$\frac{\partial P}{\partial x_j} \approx \frac{\partial}{\partial x_j} P(\nu_1, \ldots, \nu_m, x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_n)\Big|_{\substack{\nu_1 = x_1 \\ \cdots\cdots \\ \nu_m = x_m}} =$$

$$= \varkappa_j(x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_n), \qquad \text{(IX.77)}$$

$$\frac{\partial P}{\partial y_i} \approx \frac{\partial}{\partial y_i} P(x_1, \ldots, x_m, x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_n) =$$

$$= \eta_i(x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_n), \quad i = 0, 1, \ldots, n. \qquad \text{(IX.78)}$$

Proofs to this replacement of exact values of partial derivatives by their approximate values the same as in p. 2 of previous paragraph.

Page 276.

Motion to the maximum of criterion P in this case is realized according to the modified gradient method, determined by the relationships/ratios

$$\left.\begin{aligned} \dot{x}_j &= K\varkappa_j(x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_n), \quad j = 0, 1, \ldots, m; \\ \dot{y}_i &= K\eta_i(x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_n), \quad i = 0, 1, \ldots, n. \end{aligned}\right\} \quad \text{(IX.79)}$$

In accordance with formulas (IX.24) (IX.59) (IX.69) (IX.79) the schematic of the optimum self-tuning system, depicted in Fig. IX.8, is formed/shaped. The principle of the operation of this diagram is the same as the diagrams, depicted in Fig. IX.5 and IX.6.

Just as in p. 2 and 3 previous paragraphs, it is possible to show that the state of equilibrium of this self-tuning system is

reached when $x_0 = m_G$; $x_1 = \nu_1$, ...; $x_m = \nu_m$; $y_0 = m_Z$; $y_1 = \gamma_1$, ...; $y_n = \gamma_n$, the state of equilibrium of system corresponds to the maximum of criterion P in the observance of the conditions, analogous to condition (IX.55); motion to the maximum of criterion is realized monotonically.

Method proposed here of the construction of the optimum self-tuning system can be used without the fundamental changes also in the more general case, when useful signal and interference are correlated.

It is presented briefly the essence of the proposed method of the construction of the optimum self-tuning systems on different criteria.

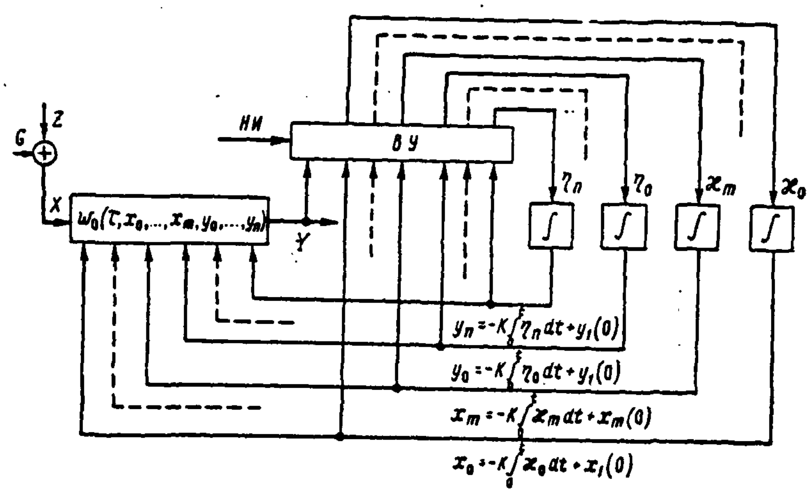



Fig. IX.8. Schematic of system with the self-adjusting according to the modified gradient method.

Page 277.

Method encompasses several stages:

First stage - finding the expression of the criterion through the weight function of fundamental control loop and through the probabilistic characteristics of useful signal and interference (through unknown parameters $\nu_j, \gamma_i$ of these probabilistic characteristics).

Second stage - obtaining the general view of the conditional optimum weight function of fundamental control loop $w_0(x_j, y_i)$ as the

functions of controlled parameters $x_j, y_i$.

The third stage - the creation of the main control loop, it is accurate (or approximately) which corresponds to conditional optimum weight function $w_0(x_j, y_i)$. This outline must contain parameters $x_j, y_i$, adjusted in the sufficiently wide limits.

The structure of fundamental control loop is determined by the form of the correlation functions of useful signal and interference, and a number of controlled parameters $x_j, y_i$ is equal to a number of unknown parameters of the probabilistic characteristics of useful signal and interferences.

Fourth stage - creation of the program of the exact or approximate computation of criterion and its gradient (or only the gradient of criterion) at current time taking into account initial and working information about the fundamental control loop and the interactions on it. With the complete initial information about the probabilistic characteristics of useful signal it is possible to accurately compute criterion and its gradient, in the absence of this initial information criterion and its gradient in principle accurately are not calculated. In the latter case it is possible to approximately determine the gradient of criterion so that the

approximate gradient method of motion to the extremum of criterion provides finding this criterion with the observance of some conditions, presented above in this chapter.

Fifth stage - creation of self-tuning loop in accordance with the calculation formulas accepted and the methods of computing the gradient of criterion.

The proposed here optimum self-tuning systems work better, the nearer to the real values of the parameters are set the initial values of parameters $x_i, y_i$ in the fundamental control loop.

Method presented above can be used also with the use of other criteria.

In self-tuning systems examined above the main control loop is linear system with the controlled parameters. Nonlinear elements of these systems - computers. In these elements/cells are realized such nonlinear devices/equipment. In these elements/cells are realized such nonlinear operations, as squaring [see formulas (IX.6) (IX.24)], multiplication, division, raising to the power [see formula (IX.24)] and so forth.

It is possible to use the methodology proposed here, also, for

constructing the self-tuning systems (including optimum) with the nonlinear fundamental control loops.

Page 278.

The task of determining of statistical criterion and its gradient substantially is complicated in these cases. The exact computation of criterion and its gradient in any complicated cases is in principle impossible. It is expedient to calculate criteria and their gradients approximately in such cases. In this case it is possible to use any linearization of nonlinear elements (systems), for example, by harmonic [75] or statistical [34] linearization.

Methodology presented here of the formation of the optimum self-tuning systems on the statistical criteria can be used both for constructing the real self-tuning systems and for the evaluation/estimate of the potential accuracy of the self-tuning systems.

Example. Let at the input of system (Fig. IX.9) operate signal X, which is the sum of useful signal G and interference Z. Let be preset the following probabilistic characteristics of interactions:

$$m_G = m_Z = 0;\ K_G = e^{-\alpha|\tau|};\ K_Z = S_Z\,\delta(\tau);$$
$$K_X = K_G + K_Z = e^{-\alpha|\tau|} + S_Z\delta(\tau).$$

where $\alpha$ - known value;

$S_z$ - unknown spectral density.

The desired output value H(t) coincides with the useful signal, i.e., H(t)=G(t). Consequently, $K_{gh} = K_g$. Integral equation (IX.60) in this case takes the form

$$\int_0^T \left[e^{-\alpha(\tau-\lambda)} + S_Z\delta(\tau-\lambda)\right] w_0(\lambda, S_Z)\, d\lambda - e^{-\alpha\tau} = 0, \quad 0 \leq \tau \leq T.$$

Let us assume T=∞ and we will seek the solution of this equation in the form [6, 60, 77]

$$w_0(\tau, S_Z) = Be^{-\beta\tau}.$$

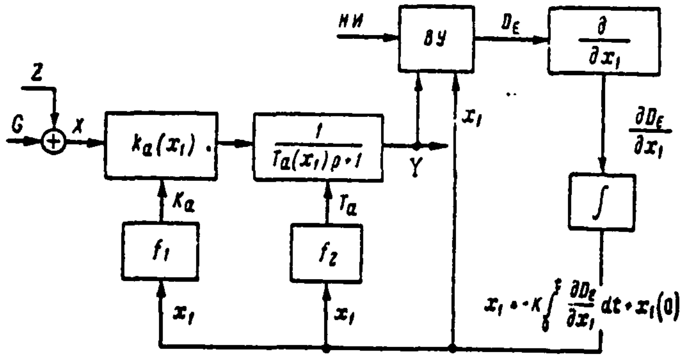



Fig. IX.9. Schematic of the system of extreme control with the weight function of the type of equation (IX.82).

Page 279.

For this let us substitute the solution, recorded in general form, in the integral equation

$$\int_0^\infty \left[e^{-\alpha|\tau-\lambda|} + S_z\delta(\tau-\lambda)\right] Be^{-\beta\lambda}d\lambda - e^{-\alpha\tau} = 0,$$
$$0 \leqslant \tau \leqslant \infty,$$

we will consider it as identity in terms of the variable $\tau$.

Fulfillment of integration gives the following result:

$$B\left(\frac{2\alpha}{\alpha^2-\beta^2}e^{-\beta\tau} + \frac{e^{-\alpha\tau}}{\alpha-\beta} + S_z e^{-\beta\tau}\right) - e^{-\alpha\tau} = 0,$$
$$0 \leqslant \tau \leqslant \infty.$$

This identity is satisfied, if are equal to zero coefficients with functions $e^{-\beta\tau}$ and $e^{-\alpha\tau}$, are made following equalities:

$$\frac{2\alpha}{\alpha^2-\beta^2} + S_z = 0; \quad \frac{B}{\alpha-\beta} - 1 = 0.$$

We easily obtain the solution of this system of equations:

$$\beta=\sqrt{\alpha\left(\alpha+\frac{2}{S_Z}\right)};\ B=\alpha-\sqrt{\alpha\left(\alpha+\frac{2}{S_Z}\right)}.$$

Thus, optimum weight function with fixed/recorded value $S_Z$ is determined by the formula

$$w_u(\tau,S_Z)\ \left[\alpha-\sqrt{\alpha\left(\alpha+\frac{2}{S_Z}\right)}\right]e^{-\sqrt{\alpha\left(\alpha+\frac{2}{S_Z}\right)}\tau}. \quad \text{(IX.80)}$$

This weight function corresponds to aperiodic component/link with the time constant $T_u$ equal to

$$\frac{1}{\sqrt{\alpha\left(\alpha+\frac{2}{S_Z}\right)}},$$

and with the factor of amplification $k_u$ equal to

$$\frac{1}{\sqrt{1+\frac{2}{S_Z\alpha}}}-1.$$

Consequently,

$$T_a=T_u(S_Z);\ k_a=k_u(S_Z). \quad \text{(IX.81)}$$

The weight function of fundamental control loop in this case depends on one variable parameter $x_1$:

$$w_0(\tau,x_1)-\left[\alpha-\sqrt{\left(\alpha+\frac{2}{x_1}\right)}\right]e^{-\sqrt{\alpha\left(\alpha+\frac{2}{x_1}\right)}\tau}. \quad \text{(IX.82)}$$

Page 280.

The corresponding outline is aperiodic component/link with the factor of amplification

$$k_a = k_a(x_1) = \frac{1}{\sqrt{1+\frac{2}{\alpha x_1}}} - 1 = f_1(x_1) \qquad \text{(IX.83)}$$

and time constant

$$T_a = T_a(x_1) = \frac{1}{\sqrt{\alpha\left(\alpha+\frac{2}{x_1}\right)}} = f_2(x_1). \qquad \text{(IX.84)}$$

Fig. IX.9 depicts diagram of the system of extreme control, fundamental control loop of which has weight function (IX.82). In the diagram for the clarity aperiodic component/link is represented in the form of two components/links: intensifying with the factor of amplification $k_a(x_1)$ and aperiodic with the amplification factor, equal to one, and with the time constant $T_a(x_1)$. Instead of the weight functions of components/links are shown their transfer functions $k_a(x_1)$ and $\frac{1}{T_a(x_1)s+1}$. Components/links with designations $f_1$ and $f_2$ are functional converters, which convert value $x_1$ according to formulas (IX.83) and (IX.84) respectively.

Page 281.

Chapter X.

SYNTHESIS OF AUTOMATIC INFORMATION-CARRYING SYSTEMS WITH NONLINEAR FILTERS.

1. Brief information from the theory of nonlinear filtration.

To research of the information-carrying systems with the different types of continuous modulation (amplitude, frequency, phase, etc.) is dedicated vast Soviet and foreign literature. Fundamental scientific-applied works in this direction it is possible to divide into three groups: production/consumption/generation and proof of expedient quantitative criteria for evaluation of quality and efficiency of systems [30, 83, 70, 112]; the analysis of different concrete/specific/actual systems and their comparison on selected criterion [22, 66, 67, 101, 102, 119]; the synthesis of the information-carrying systems [43, 126, 132, 133, 134].

Two receptions/procedures were used in mentioned and other

published works on the synthesis of the optimum information-carrying systems until recently. Informational signal is considered as the stationary random process, which possesses constant spectral density within the specific frequency band and zero out of this band.

Page 282.

This model does not reflect the property of real signals. In addition to this, the use of this model frequently leads to the misunderstandings and the paradoxes, since signal proves to be interpolated [96].

In some works to the informational signal, which is, as a rule, non-Gaussian random process, artificially were applied the known methods of the theory of the linear filtration of Kolmogorov-Wiener. In this case sometimes it was possible to obtain the correct quantitative estimations of quality (for example, signal-to-noise ratio at the output); however, a question about the structural scheme of optimum receiver remained completely unsolved. This position was explained by the absence of the worked out mathematical methods of synthesis.

In 1960 R. L. Stratonovich developed theory of the optimum nonlinear filtration of the Markov informational

communications/reports, when signal is accepted together with noise [92, 93]. This theory gives single base for determining the structural schemes of optimum receivers in connection with different signals and it makes it possible to compute the errors of filtration, which characterize the quality of the optimum methods of radio reception.

The use/application of theory of nonlinear filtration to particular radio engineering examples and some simplified receptions/procedures of the solution of problems appearing in this case were examined in the works of R. L. Stratonovich and N. K. Kuhlmann [55, 56].

The basic condition/positions of the theory of optimum nonlinear filtration are briefly given below and the used subsequently approximation method of solving the nonlinear equations of filtration is indicated. In this case it is assumed that the signal has the "stray" phase and is taken in against the background of white noise, and informational communication/report represents Markov process.

Let us assume that during certain interval of time (0, t) the input of system enters oscillation $\xi(t)$, which is the sum of signal $S(t, \lambda(t))$ and the white noise $n(t)$, i.e.

$$\xi(t) = S(t, \lambda(t)) + n(t). \qquad (X.1)$$

Here through λ(t) is designated multidimensional random vector with components $\{\lambda_1(t), \lambda_2(t), ..., \lambda_m(t)\}$, which present random processes - the parameters, on which depends the useful signal. The information interesting us can be contained not in all components of vector λ(t), but only in the part of them. The latter must be isolated from oscillation ξ(t) with the smallest error. The statistical characteristics of the white noise n(t) are considered known:

$$M[n(t)] = 0; \quad M[n(t_1)n(t_2)] = \frac{1}{2}N_0\delta(t_2 - t_1), \qquad (X.2)$$

where δ(z) - delta-function.

Page 283.

It is assumed relative to useful signal that is known the form scalar function S(t, λ(t)) from the time and the vector argument and that the multidimensional random vector

$$\lambda(t) = \lambda\{\lambda_1(t), \ldots, \lambda_m(t)\}$$

represents the Markov process, whose each component is described by the a priori stochastic differential equation of the form

$$\dot{\lambda}_\mu = c_\mu(\lambda) + n_\mu, \quad M\{n_\mu(t_1)n_\nu(t_2)\} = \frac{1}{2}N_{\mu\nu}\delta(t_2 - t_1),$$
$$\mu, \nu = 1, 2, \ldots, m; \qquad (X.3)$$

here and subsequently point on top designated time derivative.

Consequently, change in he time of the a priori probability density p(t, λ) of random vector λ(t) is determined by the equations

of Einstein-Fokker

$$\left.\begin{array}{l} \dot{p}_{pr}(t, \lambda) = L_{pr}p_{pr}(t, \lambda); \\ L_{pr} = -\sum_{\mu=1}^{m} \frac{\partial}{\partial\lambda_\mu} a_\mu + \frac{1}{4} \sum_{\mu,\nu=1}^{m} \frac{\partial^2}{\partial\lambda_\mu \partial\lambda_\nu} N_{\mu\nu}. \end{array}\right\} \quad (X.4)$$

All available information about the parameters of useful signal is contained in the a posteriori density of probability $p_{ps}(t, \lambda)$ of vector λ(t). During the computation of this a posteriori probability density two circumstances [96] must be taken into consideration. First, enumerated above a priori information and, in particular, a priori known probability law (X.4) of change in the time of process itself λ( t). In the second place, the result of observation of accepted realization ξ(t) as the previous time interval.

If we correctly fulfill appropriate mathematical computations [92, 93], then it will seem that the standardized/normalized a posteriori density of probability $p_{ps}(t, \lambda)$ of vector λ(t) must satisfy the integrodifferential equation

$$\dot{p}_{ps}(t, \lambda) = L_{pr}w_{ps}(t, \lambda) + \{F(\lambda, t) - M[F(\lambda, t)]\} p_{ps}(t, \lambda), \quad (X.5)$$

where

$$\left.\begin{array}{l} F(\lambda, t) = \frac{1}{N_0} [2S(t, \lambda)\xi(t) - S^2(t, \lambda)]; \\ M[F(\lambda, t)] = \int \ldots \int F(\lambda, t) p_{ps}(t, \lambda) d\lambda_1 \ldots d\lambda_m. \end{array}\right\} \quad (X.6)$$

Page 284.

Fundamental equation (X.5) of optimum nonlinear filtration, is

obtained by R. L. Stratonovich, determine the full/total/complete procedure of the filtration of the parameters of signal against the background of white noise. This equation is nonlinear $p_{ps}(t, \lambda)$, integrodifferential equation relative to probability density $p_{ps}(t, \lambda)$ depending on (m+1) arguments. To virtually simulate equation (X.5) is very complicated.

It is obvious that at zero time the a posteriori probability density coincides with the a priori. In proportion to processing the realization accepted occurs more exact following of the parameters of useful signal.

In the sufficiently large signal-to-noise ratios and the long time of observation is the foundations for considering that the a posteriori density of probability of vector $\lambda(t)$ will be normal. If we take this assumption and to record multidimensional normal a posteriori probability density through estimated values $\lambda_{\mu}^{*}(t)$ the components of vector $\lambda(t)$ and cumulants $K_{\mu\nu}^{*}(t)$, then from expression (X.5) for them will be obtained following equations [56]:

$$
\left.
\begin{aligned}
&\dot{\lambda}^{*}_{\mu} = c_{\mu}(\lambda^{*}) + \sum_{\nu=1}^{m} K^{*}_{\mu\nu} \frac{\partial F(\lambda^{*}, t)}{\partial \lambda^{*}_{\nu}};\\
&\dot{K}^{*}_{\mu\nu} = \frac{1}{2} N_{\mu\nu}(\lambda^{*}) + \sum_{l=1}^{m} K^{*}_{l\nu} \frac{\partial a_{\mu}(\lambda^{*})}{\partial \lambda^{*}_{l}} + \\
&+ \sum_{l=1}^{m} K^{*}_{\mu l} \frac{\partial a_{\nu}(\lambda^{*})}{\partial \lambda^{*}_{l}} + \sum_{l,j=1}^{m} K^{*}_{\mu l} K^{*}_{j\nu} \frac{\partial^{2} F(\lambda^{*}, t)}{\partial \lambda^{*}_{l} \partial \lambda^{*}_{j}}.
\end{aligned}
\right| \qquad (X.7)
$$

Optimum nonlinear system should simulate equations (X.7). At the appropriate outputs of this system must be put out optimum estimated values $\lambda^{*}_{\mu}$ (t) components $\lambda_{\mu}$ (t), filtered from the interferences in the best way. The a posteriori dispersion (error) of the filtration of component $\lambda_{\mu}$ (t) is equal to $\sigma^{2}_{\mu}(t) = K^{*}_{\mu\mu}(t)$.

After the time, which exceeds the length of transient processes by the system, the error of filtration in the large signal-to-noise ratios will fluctuate relative to its average value insignificantly. In this case the cumulants in equations (X.7) can be taken time-independent and assumed $K^{*}_{\mu\nu}(t) = K^{*}_{\mu\nu} = \mathrm{const}$ [56]. The latter are found as a result of preliminary use/application to second equation (X.7) of the operation of temporary/time averaging and subsequent determination of steady-state solution. This approximate reception/procedure will be used subsequently.

Page 285.

It is first assumed in the synthesis given below of the optimum continuous communication systems with the different types of modulation that the informational communication/report $\lambda(t)$ is described stochastic equation

$$\dot{\lambda} + \alpha\lambda = n_\lambda(t), \qquad (X.8)$$

but the unavoidable random fluctuation of the phase of signal $\varphi(t)$ is a process with independent increments [96]:

$$\dot{\varphi} = n_\varphi(t). \qquad (X.9)$$

In equations (X.8) and (X.9) $n_\lambda(t)$ and $n_\varphi(t)$ - white noises with the zero average/mean values and one-way spectral densities of $N_\lambda$ and $N_\varphi$ respectively; the parameter $\alpha$ characterizes the width of the spectrum of communication, whose a priori dispersion is equal to

$$\sigma_\lambda^2 = N_\lambda/4\alpha. \qquad (X.10)$$

By signal-to-noise ratio at the input is understood value

$$q = \frac{P}{\alpha N_0}; \quad P = \frac{1}{T}\int_0^T S^2(t, \lambda(t))\,dt \qquad (X.11)$$

$$(\omega_0 T \gg 1).$$

After one-dimensional case, which corresponds to equation (X.8), in the same sequence the signals with the different types of modulation, when informational communication/report is two-dimensional Markov process are examined.

The example to ordinary amplitude modulation is in more detail examined for the illustration of method of the use/application of theory in both cases, while computational details are omitted for other forms of modulation.

2. Synthesis of the optimum continuous communication systems for the one-component informational communications/reports.

Ordinary amplitude modulation. With the amplitude modulation (AM) the useful signal can be recorded in the form

$$S(t, \lambda) = [A_0 + M_A\lambda(t)]\cos[\omega_0 t + \varphi(t)], \quad \text{(X.12)}$$

where $A_0$ and $\omega_0$ - a priori known values of amplitude and frequency;

$\lambda(t)$ - informational parameter, defined by equation (X.8);

$\varphi(t)$ - the random phase, considered as the unessential (noninformation) parameter and determined by formula (X.9);

$M_A$ - constant coefficient.

Page 286.

In connection with this case in the gaussian approximation/approach of equation (X.7) of optimum nonlinear

filtration they take the form

$$\left.\begin{aligned} \dot{\lambda}^* &= -\alpha\lambda^* + K^*_{\lambda\lambda}F_\lambda + K^*_{\lambda\varphi}F_\varphi; \\ \dot{\varphi}^* &= K^*_{\varphi\varphi}F_\varphi + K^*_{\lambda\varphi}F_\lambda; \end{aligned}\right\} \quad (X.13)$$

$$\left.\begin{aligned} \dot{K}^*_{\lambda\lambda} &= \frac{1}{2}N_\lambda - 2\alpha K^*_{\lambda\lambda} + (K^*_{\lambda\lambda})^2 F_{\lambda\lambda} + 2K^*_{\lambda\lambda}K^*_{\lambda\varphi}F_{\lambda\varphi} + \\ &\quad + (K^*_{\lambda\varphi})^2 F_{\varphi\varphi}; \\ \dot{K}^*_{\lambda\varphi} &= -\alpha K^*_{\lambda\varphi} + K^*_{\lambda\lambda}K^*_{\lambda\varphi}F_{\lambda\lambda} + K^*_{\lambda\varphi}K^*_{\varphi\varphi}F_{\varphi\varphi} + (K^*_{\lambda\varphi})^2 F_{\lambda\varphi} + \\ &\quad + K^*_{\lambda\lambda}K^*_{\varphi\varphi}F_{\lambda\varphi}; \\ \dot{K}^*_{\varphi\varphi} &= \frac{1}{2}N_\varphi + (K^*_{\varphi\varphi})^2 F_{\varphi\varphi} + 2K^*_{\varphi\varphi}K^*_{\lambda\varphi}F_{\lambda\varphi} + (K^*_{\lambda\varphi})^2 F_{\lambda\lambda}. \end{aligned}\right\} \quad (X.14)$$

Function F is determined by formula (X.6) and it is approximately equal to

$$F = \frac{1}{N_0}\Big\{2\xi(t)\,[A_0 + M_A\lambda^*(t)]\cos[\omega_0(t) + \varphi^*(t)] - \\ - \frac{1}{2}[A_0 + M_A\lambda^*(t)]^2\Big\}.$$

Indices with function F designate derivatives from the appropriate parameter:

$$\left.\begin{aligned} F_\lambda &= \frac{\partial F}{\partial\lambda^*} = \frac{M_A}{N_0}[2\xi(t)\cos(\omega_0 t + \varphi^*) - (A_0 + M_A\lambda^*)]; \\ F_\varphi &= \frac{\partial F}{\partial\varphi^*} = -\frac{2}{N_0}[A_0 + M_A\lambda^*]\,\xi(t)\sin(\omega_0 t + \varphi^*); \\ F_{\lambda\lambda} &= \frac{\partial^2 F}{\partial\lambda^{*2}} = -\frac{M_A^2}{N_0}; \\ F_{\varphi\varphi} &= \frac{\partial^2 F}{\partial\varphi^{*2}} = -\frac{2}{N_0}[A_0 + M_A\lambda^*]\,\xi(t)\cos(\omega_0 t + \varphi^*); \\ F_{\lambda\varphi} &= \frac{\partial^2 F}{\partial\lambda^*\partial\varphi^*} = -2\frac{M_A}{N_0}\xi(t)\sin(\omega_0 t + \varphi^*). \end{aligned}\right\} \quad (X.15)$$

Keeping in mind further transition/junction to the steady state, during the recording of expressions (X.13) and (X.14) it was assumed

that $K^{\bullet}_{\lambda\varphi} = K^{\bullet}_{\varphi\lambda}$.

Page 287.

For simplification in the computation of the errors of filtration let us substitute in equations (X.14) for function $F_{\lambda\lambda}, F_{\lambda\varphi}, F_{\varphi\varphi}$ of their values, time-averaged, which let us designate by feature on top. They are equal to:

$$\overline{F}_{\lambda\lambda} = -\frac{M_A^2}{N_0}; \quad \overline{F}_{\lambda\varphi} = 0; \quad \overline{F}_{\varphi\varphi} = -\frac{1}{N_0}(A_0^2 + \sigma_A^2);$$
$$\sigma_A^2 = M_A^2\sigma_\lambda^2.$$

Furthermore, we will be bounded to the examination of steady state, i.e., let us assume $K^{\bullet}_{\lambda\lambda} = K^{\bullet}_{\lambda\varphi} = K^{\bullet}_{\varphi\varphi} = 0$. Then instead of the system of equations (X.14) we obtain

$$(\overline{K}^{\bullet}_{\lambda\lambda})^2\,\overline{F}_{\lambda\lambda} - 2\alpha\overline{K}^{\bullet}_{\lambda\lambda} + (\overline{K}^{\bullet}_{\lambda\varphi})^2\,\overline{F}_{\varphi\varphi} + \frac{1}{2}N_\lambda = 0;$$
$$\overline{K}^{\bullet}_{\lambda\varphi}(\overline{K}^{\bullet}_{\lambda\lambda}\overline{F}_{\lambda\lambda} + \overline{K}^{\bullet}_{\varphi\varphi}\overline{F}_{\varphi\varphi} - \alpha) = 0;$$
$$(\overline{K}^{\bullet}_{\varphi\varphi})^2\overline{F}_{\varphi\varphi} + (\overline{K}^{\bullet}_{\lambda\varphi})^2\overline{F}_{\lambda\lambda} + \frac{1}{2}N_\varphi = 0.$$

Hence let us find the solution of this system:

$$\overline{K}^{\bullet}_{\lambda\lambda} = \frac{\alpha N_0}{M_A^2}\left(\sqrt{1 + \frac{M_A^2 N_\lambda}{2\alpha^2 N_0}} - 1\right); \quad \overline{K}^{\bullet}_{\varphi\varphi} = \sqrt{\frac{N_0 N_\varphi}{2(A_0^2 + \sigma_A^2)}}; \quad \overline{K}^{\bullet}_{\lambda\varphi} = 0 \tag{X.16}$$

We will characterize the freedom from interference of the reception of continuous signals with the value of the relative error of the filtration of informational communication/report. The square

of this error is determined by the formula

$$\delta_{AM}^2 = \frac{\overline{K}^*_{\lambda\lambda}}{\sigma_\lambda^2} = \frac{\alpha N_0}{\sigma_A^2}\left(\sqrt{1+2\frac{\sigma_A^2}{\alpha N_0}}-1\right). \qquad (X.17)$$

Let us introduce here signal-to-noise ratio:

$$q = \frac{P}{\alpha N_0} = \left(\frac{\sigma_A^2}{2\alpha N_0}\right)\frac{1+m^2}{m^2}; \quad m = \frac{\sigma_A}{A_0}. \qquad (X.18)$$

Then formula (X.17) for the square of the relative error of filtration can be recorded in the following final form:

$$\delta_{AM}^2 = \frac{1}{2q\frac{m^2}{1+m^2}}\left(\sqrt{1+4q\frac{m^2}{1+m^2}}-1\right). \qquad (X.19)$$

It is characteristic that in the approximation/approach examined the relative error of the optimum filtration of the amplitude-modulated signals does not depend on the value of the phase fluctuations of signal, i.e., the freedom from interference of coherent ($N_\varphi = 0$) and quasi-coherent ($N_\varphi \neq 0$) reception/procedure proves to be identical.

Page 288.

After supplying in equations (X.13) the obtained average/mean values of cumulants (X.16) and the first two expressions from system (X.15), we will obtain

$$\left.\begin{aligned} \dot{\lambda}^{*} + \alpha'\lambda^{*} &= 2\overline{K}^{*}_{\lambda\lambda}\frac{M_A}{N_0}\left[\xi(t)\cos(\omega_0 t+\varphi^{*})-\frac{1}{2}A_0\right];\\ \alpha' &= \alpha\left(1+2\frac{\sigma_A^2}{\alpha N_0}\right)^{1/2};\\ \dot{\varphi}^{*} &= -\frac{2}{N_0}(A_0+M_A\lambda^{*})\,\overline{K}^{*}_{\varphi\varphi}\xi(t)\sin(\omega_0 t+\varphi^{*}). \end{aligned}\right| \qquad (X.20)$$

These equations are simulated by the optimum receiver, one of the versions of structural scheme of which is given in Fig. X.1. On the given diagram: ПГ - adjustable/tuneable high-frequency oscillator; УЭ - control device.

Optimum receiver realizes quasi-coherent reception of signals. It has the fundamental, informational channel, at output of which is obtained estimated the value $\lambda*(t)$, and the phase automatic frequency control, which develops reference signal.

The characteristic feature of diagram lies in the fact that amplification factor

$$K = 2\overline{K}^{*}_{\lambda\lambda}\frac{M_A}{N_0} = \frac{2\alpha}{M_A}\left(\sqrt{1+\frac{2\sigma_A^2}{\alpha N_0}}-1\right)$$

amplifier and its time constant

$$T' = \frac{1}{\alpha'} = \frac{1}{\alpha}\left(1+\frac{2\sigma_A^2}{\alpha N_0}\right)^{-1/2}$$

depend on spectral density $N_0$ of additive white noise.

At the unknown value of coefficient of $N_0$ or with its change with time according to the previously not provided law for the realization of optimum reception/procedure is necessary special device/equipment for measurement $N_0$. In this case the diagram will be that self-tuning.

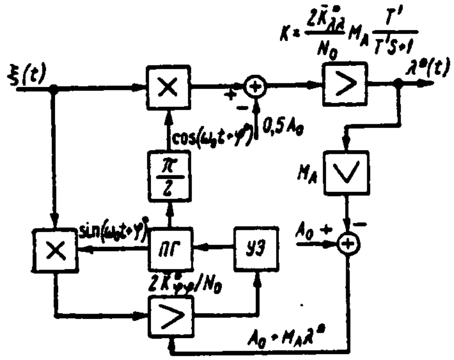



Fig. X.1. Structural scheme of the optimum receiver of the amplitude-modulated radio signals.

Page 289.

This also relates to the diagram of phase automatic frequency control, since voltage/stress $A_0+M_A\lambda^*(t)$ influences to the amplifier in the feedback loop as the signal ARU.

Two-band modulation without the carrier. Let us record two-band signal without the carrier (DM) in the form

$$S(t,\lambda)\quad M_A\lambda(t)\cos[\omega_0 t+\varphi(t)]. \qquad (X.21)$$

From the comparison of the form of useful signals (X.12) and (X.21) it directly follows that for the signal in question are valid all obtained above expressions, it is necessary only in them to assume $A_0=0$. In particular, it is not difficult to ascertain that now

will be valid the following relationships/ratios:

$$F_\lambda = \frac{M_A}{N_0}[2\xi(t)\cos(\omega_0 t + \varphi^*) - M_A\lambda^*];$$
$$F_\varphi = -\frac{2}{N_0}M_A\lambda^*(t)\,\xi(t)\sin(\omega_0 t + \varphi^*);$$
$$\overline{K}^*_{\lambda\lambda} = \frac{\alpha N_0}{M_A^2}\left(\sqrt{1 + \frac{M_A^2 N_\lambda}{2\alpha^2 N_0}} - 1\right);$$
$$\overline{K}^*_{\varphi\varphi} = \frac{1}{\sigma_A}\sqrt{\frac{1}{2}N_0 N_\varphi}; \quad \overline{K}^*_{\lambda\varphi} = 0.$$

For the square of the relative error of the filtration of communication/report with two-band modulation without the carrier we will obtain the formula

$$\delta^2_{ДМ} = \frac{1}{2q}(\sqrt{1+4q} - 1); \quad q = \frac{\sigma_A^2}{2\alpha N_0}. \qquad \text{(X.22)}$$

In connection with signal (X.21) equations (X.20) take the form

$$\left.\begin{aligned} \dot{\lambda}^* + \alpha\lambda^* &= 2\overline{K}^*_{\lambda\lambda}\frac{M_A}{N_0}\xi(t)\cos(\omega_0 t + \varphi^*); \\ \dot{\varphi}^* &= -\frac{2}{N_0}M_A\lambda^*\overline{K}^*_{\varphi\varphi}\xi(t)\sin(\omega_0 t + \varphi^*). \end{aligned}\right\} \qquad \text{(X.23)}$$

The structural scheme of optimum receiver, constructed in accordance with equations (X.23), is depicted in Fig. X.2.

Single-band modulation with the pilot signal. Useful radio signal in single-band modulation (OM) and presence of the pilot signal, which contains information about the phase, can be represented in the following form [124]:

$$S(t, \lambda) = \frac{1}{\sqrt{2}} M_A \lambda(t) \cos[\omega_0 t + \varphi(t)] + \\ + \frac{1}{\sqrt{2}} M_A \hat{\lambda}(t) \sin[\omega_0 t + \varphi(t)] + C_0 \cos[\omega_0 t + \varphi(t)], \quad (X.24)$$

where $C_0$ - known constant value of the amplitude of pilot signal;

$\lambda(t)$ - the transformation of Gilbert/Hilbert from informational communication $\lambda(t)$;

$$\hat{\lambda}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\lambda(t-\tau)}{\tau} d\tau.$$

Page 290.

Single-band signal is represented in the form of the sum of two two-band signals. The four-phase method of shaping of single-band signals [19] corresponds to expression (X.24).

Let us note [124, 129] that the transformation of Gilbert/Hilbert from the normal stationary process is also normal stationary process with the same autocorrelation function, moreover both processes are mutually not correlated at the coinciding moments of time. As a result of the double use/application of transformation of Gilbert/Hilbert is obtained initial process with the opposite sign (for example, $\hat{\hat{n}}(t)=-n(t)$).

The pilot signal, entering expression (X.24), is used for the work of phase self-alignment of frequency (FAPCh), forming the supporting/reference harmonic oscillation, with the help of which is realized the quasi-coherent reception/procedure of informational radio signal. On the basis of the practical considerations of the guarantee of the best work of FAPCh is expediently to preliminarily filter out pilot signal from the informational radio signal via of inclusion/connection at the input FAPCh the sufficiently narrow-band oscillatory circuit, tuned for the accurately known frequency $\omega_0$. If outline causes systematic phase shift, then it is possible to compensate it by the start of phase shifter.

Bearing in mind that in the approximation/approach of the fluctuation of the phase of supporting/reference oscillation in question they do not affect the error of the filtration of communication/report, we will consider that the input of the channel of the formation of supporting/reference oscillation enters the sum of pilot signal and white noise:

$$\eta(t) \approx C_0 \cos[\omega_0 t + \varphi(t)] + n(t). \qquad \text{(X.25)}$$

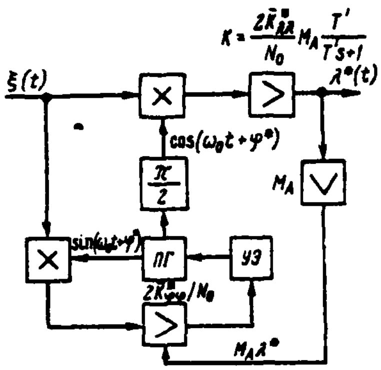



Fig. X.2. Structural scheme of the optimum receiver of two-band radio signal without oscillation of the carrier frequency with the amplitude modulation.

Page 291.

Let us assume that the random parameters of single-band signal are described by a priori stochastic equations (X.8) and (X.9) and by the further equation

$$\dot{\hat{\lambda}}+\alpha\hat{\lambda}=\hat{n}_\lambda(t), \qquad \text{(X.26)}$$

which is obtained as a result of applying the operation of the transformation of Gilbert/Hilbert to equation (X.8), i.e., here $\hat{n}_\lambda(t)$ - Gilbert/Hilbert's transformation from white noise $n_\lambda(t)$.

If necessary or for the mathematical correctness white noise $n_\lambda(t)$ can be considered as the normal stationary process, which has the

final, but very short time of correlation $\tau_{\kappa} \ll 1/\alpha$ [96].

If we use the previous procedure of simplification in the expressions for the errors of filtration (namely, to substitute in the corresponding equations the time-averaged expressions of function $F_{ij}$ and then to switch over to steady state), then let us arrive at the following system of equations of the optimum nonlinear filtration:

$$\left.\begin{aligned} \dot{\lambda}^* + \alpha\lambda^* &= \overline{K}^*_{\lambda\lambda} F_{\lambda}; \\ \dot{\hat{\lambda}}^* + \alpha\hat{\lambda}^* &= \overline{K}^*_{\lambda\lambda} F_{\hat{\lambda}}; \\ \dot{\varphi} &= \overline{K}^*_{\varphi\varphi} \Phi_{\varphi}, \end{aligned}\right\} \qquad (X.27)$$

where

$$\left.\begin{gathered} F = \frac{1}{N_0}\left[2\xi(t) S(t, \lambda^*) - S^2(t, \lambda^*)\right]; \\ \Phi = \frac{2}{N_0} C_0 \eta(t) \cos(\omega_0 t + \varphi^*); \\ F_{\lambda} = \frac{M_A}{N_0}\left[\sqrt{2}\,\xi(t)\cos(\omega_0 t + \varphi^*) - \frac{1}{2} M_A \lambda^* - \frac{1}{\sqrt{2}} C_0\right]; \\ F_{\hat{\lambda}} = \frac{M_A}{N_0}\left[\sqrt{2}\,\xi(t)\sin(\omega_0 t + \varphi^*) - \frac{1}{2} M_A \hat{\lambda}^*\right]; \\ \Phi_{\varphi} = -\frac{2}{N_0} C_0 \eta(t) \sin(\omega_0 t + \varphi^*); \quad \overline{K}^*_{\varphi\varphi} = \sqrt{\frac{N_0 N_{\varphi}}{2C_0^2}}; \\ \overline{K}^*_{\lambda\lambda} = \overline{K}^*_{\hat{\lambda}\hat{\lambda}} = \frac{2\alpha N_0}{M_A^2}\left(\sqrt{1 + \frac{M_A^2 N_{\lambda}}{4\alpha^2 N_0}} - 1\right); \\ \overline{K}^*_{\lambda\hat{\lambda}} = 0. \end{gathered}\right\} \qquad (X.28)$$

Signal-to-noise ratio at the input can be represented in the form

$$q = \frac{P}{\alpha N_0} = \frac{\sigma_A^2}{2\alpha N_0}\left(\frac{1+m_0^2}{m_0^2}\right), \qquad (X.29)$$

where $m_0^2 = \sigma_A^2/C_0^2$ - coefficient of division of power between the informational signal and the pilot signal.

Page 292.

After substituting into the formula for $\bar{K}^*_{\lambda\lambda}$ expression $\sigma_\lambda^2$ and $N_\lambda = 4\alpha\sigma_\lambda^2$ and after introducing relationship/ratio (X.29), it is possible to write

$$\bar{K}^*_{\lambda\lambda} = \frac{\sigma_\lambda^2}{q\dfrac{m_0^2}{1+m_0^2}}\left[\sqrt{1+2q\frac{m_0^2}{1+m_0^2}}-1\right]. \qquad (X.30)$$

Equations (X.27) after the substitution in them of obtained expressions (X.28) take the form

$$\left.\begin{aligned}
&\dot{\lambda}^* + \alpha\lambda^*\sqrt{1+\frac{\sigma_A^2}{\alpha N_0}} = \frac{2\alpha\sqrt{2}}{M_A}\left(\sqrt{1+\frac{\sigma_A^2}{\alpha N_0}}-1\right)\times\\
&\qquad\times\left[\xi(t)\cos(\omega_0 t+\varphi^*)-\frac{1}{2}C_0\right];\\
&\dot{\hat{\lambda}}^* + \alpha\hat{\lambda}^*\sqrt{1+\frac{\sigma_A^2}{\alpha N_0}} = \frac{2\alpha\sqrt{2}}{M_A}\left(\sqrt{1+\frac{\sigma_A^2}{\alpha N_0}}-1\right)\times\\
&\qquad\times\xi(t)\sin(\omega_0 t+\varphi^*);\\
&\dot{\varphi}^* = -\sqrt{\frac{2N_\varphi}{N_0}}\,\eta(t)\sin(\omega_0 t+\varphi^*).
\end{aligned}\right\} \qquad (X.31)$$

Fig. X.3 gives the structural scheme of the optimum receiver, which simulates equations (X.31). On the given diagram component/link H realizes an operation of the transformation of Gilbert/Hilbert above process of 1/2 $\hat{\lambda}*(t)$. Besides the channel of synchronization, the receiver has two informational channels.

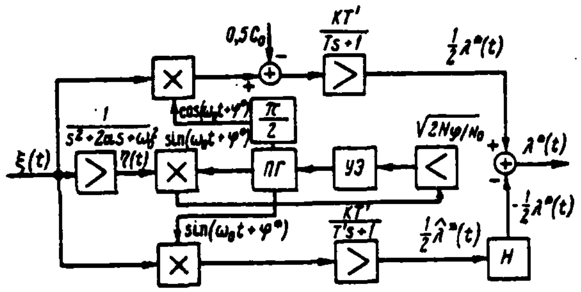



Fig. X.3. Structural scheme of the optimum receiver of single-band radio signals with the amplitude modulation.

Page 293.

The amplifier gains are selected so that at the output of the first channel is allotted estimated value 1/2 λ*(t), and at the output of the second it is reproduced by 1/2 $\hat{\lambda}$*(t). Let us designate the instantaneous values of the errors of the filtration of first and second channels respectively through 1/2 $\epsilon_1$(t) and 1/2 $\epsilon_2$(t), i.e.

$$\lambda(t) - \lambda^*(t) = \varepsilon_1(t); \quad \hat{\lambda}(t) - \hat{\lambda}^*(t) = \varepsilon_2(t), \qquad (X.32)$$

where

$$M\left[\overline{\varepsilon_1^2(t)}\right] = M\left[\overline{\varepsilon_2^2(t)}\right] = \bar{K}^*_{\lambda\lambda}. \qquad (X.33)$$

Since the given examination (gaussian approximation/approach) it is correct for the large signal-to-noise ratios ($q \gg 1$; $\sigma_\lambda^2 \gg \alpha N_0$), then it is possible to place sin (φ-φ*)≈0; cos (φ-φ*)≈1. In this case

first two equations (X.31) after rejection in the right sides of the members, who contain the harmonic multipliers of double frequency, are simplified:

$$\left.\begin{aligned}\dot{\lambda}^* + \alpha'\lambda^* &= (\alpha' - \alpha)\,\lambda(t) + \frac{2\sqrt{2}}{M_A}(\alpha' - \alpha)\,n_c(t);\\ \dot{\hat{\lambda}}^* + \alpha'\hat{\lambda}^* &= (\alpha' - \alpha)\,\hat{\lambda}(t) + \frac{2\sqrt{2}}{M_A}(\alpha' - \alpha)\,n_s(t),\end{aligned}\right\} \quad (X.34)$$

where

$$n_c(t) = n(t)\cos[\omega_0 t + \varphi^*(t)];$$
$$n_s(t) = n(t)\sin[\omega_0 t + \varphi^*(t)]; \quad \alpha' = \alpha\sqrt{1 + \frac{\sigma_A^2}{\alpha N_0}}. \quad (X.35)$$

If we into equations (X.34) substitute $\lambda*(t)=\lambda(t)-\epsilon_1(t)$ and $\hat{\lambda}*(t)=\hat{\lambda}(t)-\epsilon_2(t)$, and then to take into account equalities (X.8) and (X.26), then for the instantaneous errors of filtration we will obtain the differential equations

$$\left.\begin{aligned}\dot{\varepsilon}_1 + \alpha'\varepsilon_1 &= n_\lambda(t) - \frac{2\sqrt{2}}{M_A}(\alpha' - \alpha)\,n_c(t);\\ \dot{\varepsilon}_2 + \alpha'\varepsilon_2 &= \hat{n}_\lambda(t) - \frac{2\sqrt{2}}{M_A}(\alpha' - \alpha)\,n_s(t).\end{aligned}\right\} \quad (X.36)$$

In the previous examination the random processes $\lambda(t)$ and $\hat{\lambda}(t)$ were considered as independent. However, only informational communication/report $\lambda(t)$ interests us. Therefore should be reasonably ordered the information, which is contained at the output of second channel, which is unambiguously connected with the communication/report $\lambda(t)$.

Page 294.

For this let us join outputs of both channels as follows. First, let us use to the output signal 1/2 $\hat{\lambda}*(t)$ transformation of Gilbert/Hilbert, as a result of which we will obtain $\frac{1}{2}\hat{\hat{\lambda}}^*(t)=-\frac{1}{2}\lambda(t)-\frac{1}{2}\hat{\varepsilon}_2(t)$. In the second place, let us subtract the obtained result from the output signal of the first channel. Thus, the output signal of receiver will be equal to

$$\Lambda^*=\frac{1}{2}\lambda^*(t)-\frac{1}{2}\hat{\hat{\lambda}}^*(t)=\lambda(t)+\frac{1}{2}\left[\hat{\varepsilon}_2(t)-\varepsilon_1(t)\right]; \quad (X.37)$$

Hence we obtain expression for the instantaneous error of the filtration

$$\Lambda^*(t)-\lambda(t)=\frac{1}{2}\left[\hat{\varepsilon}_2(t)-\varepsilon_1(t)\right]. \quad (X.38)$$

Variance of error, obviously, is equal to

$$M[\overline{(\Lambda^*-\lambda)^2}]=\frac{1}{4}\{M\,\overline{[\varepsilon_1^2(t)]}+M\,\overline{[\hat{\varepsilon}_2^2(t)]}-2M\,\overline{[\varepsilon_1(t)\hat{\varepsilon}_2(t)]}\}=$$
$$=\frac{1}{2}\left[\bar{K}^*_{\lambda\lambda}-M\,\overline{[\varepsilon_1(t)\hat{\varepsilon}_2(t)]}\right]. \quad (X.39)$$

After using to second equation (X.36) the transformation of Gilbert/Hilbert, we will obtain

$$\dot{\hat{\varepsilon}}_2+\alpha'\hat{\varepsilon}_2=-n_\lambda(t)-\frac{2\sqrt{2}}{M_A}(\alpha'-\alpha)\,\hat{n}_s(t). \quad (X.40)$$

From first equation (X.36) and equation (X.40) we have

$$\left.\begin{aligned}\varepsilon_1(t) &= e^{-\alpha' t}\int\limits_{-\infty}^{t} e^{\alpha' x}\left[n_\lambda(x)-\frac{2\sqrt{2}}{M_A}(\alpha'-\alpha)\,n_c(x)\right]dx;\\ \hat{\varepsilon}_2(t) &= -e^{-\alpha' t}\int\limits_{-\infty}^{t} e^{\alpha' y}\left[n_\lambda(y)+\frac{2\sqrt{2}}{M_A}(\alpha'-\alpha)\,\hat{n}_s(y)\right]dy.\end{aligned}\right\} \quad (X.41)$$

It is clear that random processes $n_c$ (x) and $\hat{n}_s$ (y) are not correlated with $n_\lambda$ (z). In addition to this, processes $n_c$(x) and $\hat{n}_s$ (y) are also not correlated:

$$M\,[n_c(x)\,\hat{n}(y)] = \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{1}{\tau}M\,[n_c(x)\,n_s(y-\tau)]\,d\tau = 0. \quad (X.42)$$

Page 295.

Actually/really,

$$\begin{aligned}M\,[n_c(x)\,n_s(y-\tau)] &= M\,[n(x)\,n(y-\tau)]\,M\,[\cos(\omega_0 x+\varphi^*(x))\ \sin\times\\ &\quad\times(\omega_0 y-\omega_0\tau+\varphi^*(y-\tau))] =\\ &\frac{1}{2}N_0\delta(x+\tau-y)\,M\,[\cos(\omega_0 x+\varphi^*(x))\sin(\omega_0 x+\varphi^*(x))] =\\ &= \frac{1}{4}N_0\delta(x+\tau-y)\,M\,[\sin 2(\omega_0 x+\varphi^*(x))].\end{aligned}$$

Under conditions indicated earlier the random phase $\varphi^*(t)$ can be considered evenly distributed in the interval $(-\pi, \pi)$. Therefore

$$M\,[\sin 2(\omega_0 x+\varphi^*(x))] = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\sin 2(\omega_0 x+\varphi^*)\,d\varphi^* = 0.$$

Consequently, on the basis of expressions (X.41) we find

$$M[e_1(t)\hat{e}_2(t)] = -e^{-2\alpha' t}\int_{-\infty}^{t}\int_{-\infty}^{t} e^{\alpha'(x+y)} M[n_\lambda(x)\, n_\lambda(y)]\, dx\, dy =$$

$$= -\frac{N_\lambda}{4\alpha'} = -\frac{\sigma_\lambda^2}{\sqrt{1+\frac{\sigma_A^2}{\alpha N_0}}}. \qquad (X.43)$$

After substituting into expression (X.39) single values from relationships/ratios (X.30) and (X.43), we obtain final formula for the square of the relative error of the filtration:

$$\delta_{OM}^2 = \frac{1}{\sigma_\lambda^2} M\,[\overline{(\Lambda^* - \lambda)^2}] = \frac{1}{2q\frac{m_0^2}{1+m_0^2}} \times$$

$$\times\left[\sqrt{1+2q\frac{m_0^2}{1+m_0^2}} + \frac{q\frac{m_0^2}{1+m_0^2}}{\sqrt{1+2q\frac{m_0^2}{1+m_0^2}}} - 1\right]. \qquad (X.44)$$

Fig. X.4 presents the results of calculations according to formulas (X.19) (X.22) and (X.44) the square of the relative error of the filtration of informational communication/report for forms examined above of amplitude modulation.

Page 296.

It follows from the represented graphs and directly from the indicated formulas that two-band modulation with respect to freedom from interference has an advantage over other forms of amplitude

modulation. However, single-band radio signal has the narrower spectrum.

Phase modulation. Let the useful radio signal adopted with phase modulation (FM) take the form

$$S(t, \lambda) = A_0 \cos[\omega_0 t + \theta(t)];$$
$$\theta(t) = \vartheta(t) + \varphi(t);$$
$$\vartheta(t) = M_\Phi \lambda(t), \quad (X.45)$$

moreover the character of a change in the random phase $\theta(t)$ is described by the following a priori stochastic equations:

$$\theta = -\alpha M_\Phi \lambda + M_\Phi n_\lambda(t) +$$
$$+ n_\varphi(t);$$
$$\dot{\lambda} = -\alpha\lambda + n_\lambda(t). \quad (X.46)$$

Here $\lambda(t)$ - informational communication/report; $\alpha$ and $M_\Phi$ - constant coefficients, and independent noises $n_\lambda$ (t) and $n_\varphi$ (t) make sense indicated earlier.

In connection with signal (X.45) of the equation of filtration, determining structurally the schematic of optimum receiver, take the form

$$\left.\begin{array}{c} \lambda^* = -\alpha\lambda^* + \overline{K}^*_{\lambda\theta} F_\theta; \quad \dot{\theta} = -\alpha M_\Phi \lambda^* + \overline{K}^*_{\theta\theta} F_\theta; \\ F_\theta = -\dfrac{2}{N_0} A_0 \xi(t) \sin(\omega_0 t + \theta^*). \end{array}\right\} \quad (X.47)$$

As a result of solving the nonlinear system of equations for the

errors of filtration we will obtain the formulas

$$\left.\begin{aligned}\bar{K}^{\bullet}_{\lambda\lambda} = \frac{\sigma^2_\lambda}{2q\sigma^2_\Phi}(1+\sqrt{2qD_\varphi})\left[\sqrt{(1+\sqrt{2qD_\varphi})^2+4q\sigma^2_\vartheta} - \right.\\ \left. -(1+\sqrt{2qD_\varphi})\right];\end{aligned}\right\} \quad (X.48)$$

$$\left.\begin{aligned}\bar{K}^{\bullet}_{\theta\theta} = \frac{1}{2q}\left[\sqrt{(1+\sqrt{2qD_\varphi})^2+4q\sigma^2_\vartheta} - 1\right];\\ \bar{K}^{\bullet}_{\lambda\theta} = \frac{\sigma_\lambda}{2q\sigma_\vartheta}\left[\sqrt{(1+\sqrt{2qD_\varphi})^2+4q\sigma^2_\vartheta} - (1+\sqrt{2qD_\varphi})\right],\end{aligned}\right\} \quad (X.48)$$

where $$q = \frac{A_0^2}{2\alpha N_\theta};\quad \sigma^2_\Phi = M^2_\Phi\sigma^2_\lambda = M^2_\Phi\frac{N_\lambda}{4\alpha};\quad D_\varphi = \frac{1}{2\alpha}N_\varphi; \quad (X.49)$$

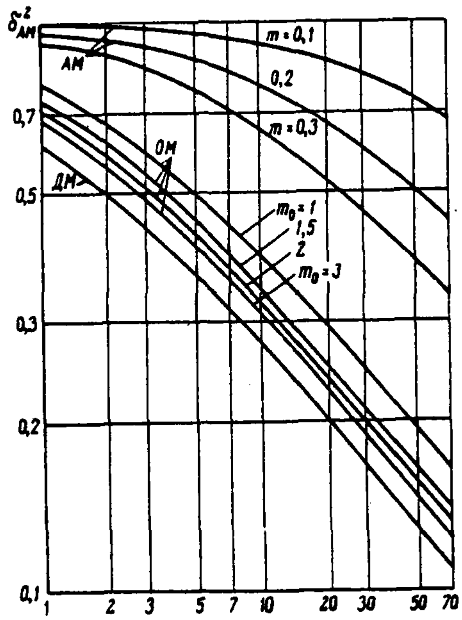



Fig. X.4. Dependence of the square of the relative error of the filtration of communication/report on the signal-to-noise ratio with the different types of amplitude modulation.

Page 297.

$D_\varphi$ is dispersion of random phase change of high-frequency oscillation for the time of the correlation of communication/report $\tau_c = 1/\alpha$.

The version of the structural scheme of the optimum (quasi-coherent) receiver, which simulates equations (X.47), is given

in Fig. X.5. Receiver is actually the follower, which realizes phase tracking of the radio signal adopted.

From first formula (X.48) we find the square of the relative error of the filtration of informational communication/report with phase modulation:

$$\delta^2_{\Phi M} = \frac{\overline{K}^{\bullet}_{\lambda\lambda}}{\sigma^2_{\lambda}} = \frac{1}{2q\sigma^2_{\theta}}(1+\sqrt{2qD_{\varphi}})\left[\sqrt{(1+\sqrt{2qD_{\varphi}})^2+4q\sigma^2_{\theta}} - \right.$$
$$\left. -(1+\sqrt{2qD_{\varphi}})\right]. \quad \text{(X.50)}$$

The results of computations according to this formula for several values $\sigma_{\theta}$ and $D_{\varphi}$ are represented graphically in Fig. X.6. It is evident from the graphs that for the given values of q and $\sigma_{\theta}$ the error of filtration is minimum when $D_{\varphi}=0$ and it increases with increase/growth $D_{\varphi}$. Therefore even with phase modulation it is expedient to raise frequency stability of oscillator.

At the fixed values of q and $D_{\varphi}$ the error of filtration depends substantially on $\sigma_{\theta}$, moreover error is reduced with increase $\sigma_{\theta}$. However, it is necessary to keep in mind that with increase $\sigma_{\theta}$ the spectrum of radio signal is expanded. In connection with the specific conditions, when signal-to-noise ratio at the input is preset, value $\sigma_{\theta}$ one should choose on the basis of the trade-off: obtaining least possible error with the minimally permissible width of the spectrum of signal.

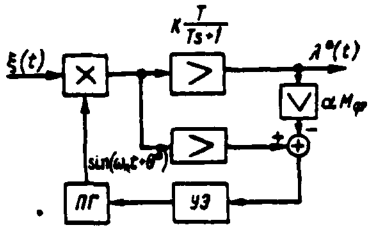



Fig. X.5. Structural scheme of the optimum receiver of the phase-modulated radio signals.

Page 298.

Frequency modulation. Let us consider the example to the frequency modulation (ChM), when useful radio signal takes the form

$$S(t, \lambda) = A_0 \cos[\omega_0 t + \psi(t)], \qquad (X.51)$$

where

$$\psi(t) = \varphi(t) + M_{\text{ч}} \int_0^t \lambda(\tau)\, d\tau;$$

$$\dot{\psi} = M_{\text{ч}}\lambda + n_{\varphi}(t).$$

For radio signal (X.51) we will obtain the following equations of optimum filtration, in accordance with which must be constructed receiver circuit.

$$\left.\begin{aligned} \dot{\lambda}^* &= -\alpha\lambda^* + \overline{K}^*_{\lambda\psi}F_{\psi}; \quad \dot{\psi} = M_{\text{ч}}\lambda^* + \overline{K}^*_{\psi\psi}F_{\psi}; \\ F_{\psi} &= -\frac{2}{N_0} A_0 \xi(t) \sin(\omega_0 t + \psi^*). \end{aligned}\right\} \qquad (X.52)$$

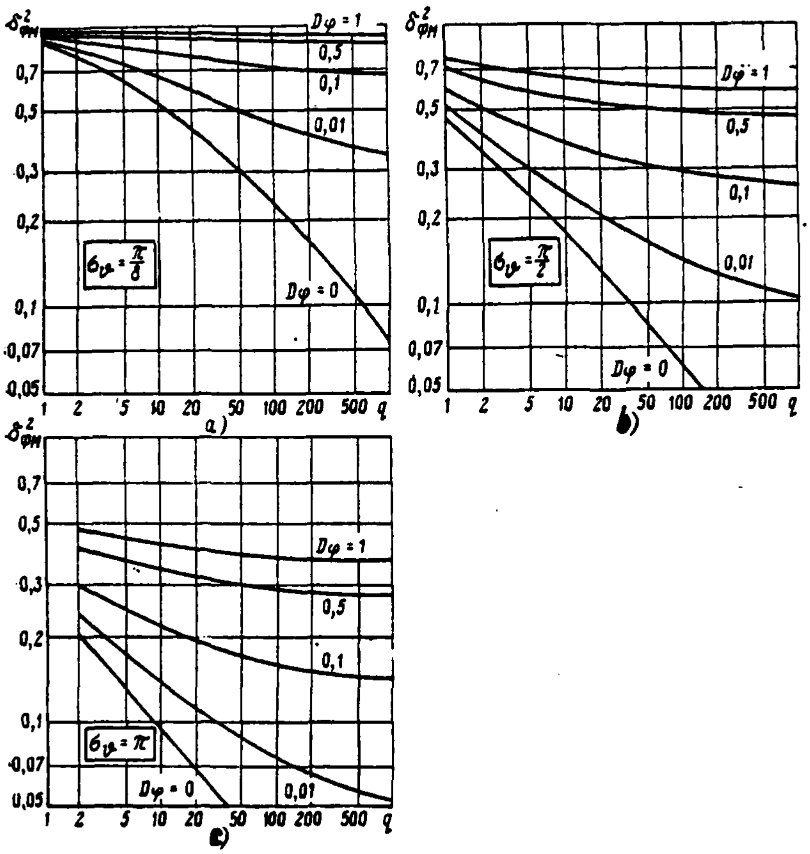



Fig. X.6. Dependence of the square of the relative error of the filtration of communication/report on the signal-to-noise ratio with phase modulation.

Page 299.

Without writing out the system of nonlinear equations for the errors of filtration, let us give the results of its solution:

$$\left.\begin{aligned}
\bar{K}^{\bullet}_{\lambda\lambda} &= \frac{\sigma_\lambda^2}{2q\beta^2_{\text{ч}M}}\left[1+2\sqrt{q\left(\beta^2_{\text{ч}M}+\frac{1}{2}D_\varphi\right)}-\right.\\
&\left.-\sqrt{1+2qD_\varphi+4\sqrt{q\left(\beta^2_{\text{ч}M}+\frac{1}{2}D_\varphi\right)}}\right]\times\\
&\times\sqrt{1+2qD_\varphi+4\sqrt{q\left(\beta^2_{\text{ч}M}+\frac{1}{2}D_\varphi\right)}};\\
\bar{K}^{\bullet}_{\psi\psi} &= \frac{1}{2q}\left(\sqrt{1+2qD_\varphi+4\sqrt{q\left(\beta^2_{\text{ч}M}+\frac{1}{2}D_\varphi\right)}}-1\right),\\
\bar{K}^{\bullet}_{\lambda\psi} &= \frac{\sigma_\lambda}{2q\beta_{\text{ч}M}}\left(1+2\sqrt{q\left(\beta^2_{\text{ч}M}+\frac{1}{2}D_\varphi\right)}-\right.\\
&\left.-\sqrt{1+2qD_\varphi+4\sqrt{q\left(\beta^2_{\text{ч}M}+\frac{1}{2}D_\varphi\right)}}\right).
\end{aligned}\right\} \quad (X.53)$$

Here $q=A_0^2/2\alpha N_0$ - signal-to-noise ratio is at the input;

$\beta_{\text{ч}M} = \sigma_\omega/\alpha = M_{\text{ч}}\sigma_\lambda/\alpha$ — the "index" of frequency modulation.

The structural scheme of the optimum filtering device/equipment, comprised according to equations (X.52), is depicted in Fig. X.7. Actually it realizes the quasi-coherent perfecting of the oscillation

accepted.

From first expression (X.53) we obtain formula for the square of the relative error of the filtration:

$$\delta^2_{\text{чм}} = \frac{\overline{K}^*_{\lambda\lambda}}{\sigma^2_\lambda} = \frac{1}{2q\beta^2_{\text{чм}}}\left[1 + 2\sqrt{q\left(\beta^2_{\text{чм}} + \frac{1}{2}D_\varphi\right)} - \right.$$
$$\left. - \sqrt{1 + 2qD_\varphi + 4\sqrt{q\left(\beta^2_{\text{чм}} + \frac{1}{2}D_\varphi\right)}}\,\right] \times$$
$$\times \sqrt{1 + 2qD_\varphi + 4\sqrt{q\left(\beta^2_{\text{чм}} + \frac{1}{2}D_\varphi\right)}}. \qquad \text{(X.54)}$$

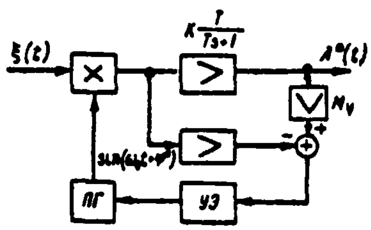



Fig. X.7. Structural scheme of the optimum receiver of the frequency modulated radio signals.

Page 300.

Fig. X.8 depicts the results of computations according to this formula for several values $\beta_{чм}$ and $D_\varphi$. It is evident from the graphs that at the fixed values of q and $\beta_{чм}$ the error of filtration is minimum, when $D_\varphi = 0$. With preset q and $D_\varphi$ the error is reduced with an increase in the "index" of modulation $\beta_{чм}$. However, in this case the spectrum of radio signal is expanded. Therefore in each specific practical case should be chosen compromise values $\beta_{чм}$ on the basis of the required accuracy of reproduction of communication/report and permissible band of frequencies of the radio channel.

If we equate expressions (X.50) and (X.54), then it is possible to find corres. ones $\sigma_\theta$ and $\beta_{чм}$, for which the freedom from interference of the optimum reception of the phase- and frequency-modulated signals is identical. In particular, when $D_\varphi = 0$

we will obtain

$$\sigma_\Phi^2 = \frac{\beta_{чM}^2}{1 + 4\sqrt{q}\,\beta_{чM}}. \qquad \text{(X.55)}$$

In this case correlation functions (X.45) and (X.51) of the FM and ChM signals are respectively equal to:

$$\begin{aligned} K_{\Phi M}(\tau) &= \frac{1}{2} A_0^2 \rho_{\Phi M}(\tau) \cos\omega_0\tau; \\ K_{чM}(\tau) &\doteq \frac{1}{2} A_0^2 \rho_{чM}(\tau) \cos\omega_0\tau, \; (D_\varphi = 0), \end{aligned} \qquad \text{(X.56)}$$

where

$$\left.\begin{aligned} \rho_{\Phi M}(\tau) &= \exp\left[-\sigma_\Phi^2 \left(1 - e^{-\alpha|\tau|}\right)\right]; \\ \rho_{чM}(\tau) &= \exp\left[-\beta_{чM}^2 \left(\alpha|\tau| - 1 + e^{-\alpha|\tau|}\right)\right]. \end{aligned}\right\} \qquad \text{(X.57)}$$

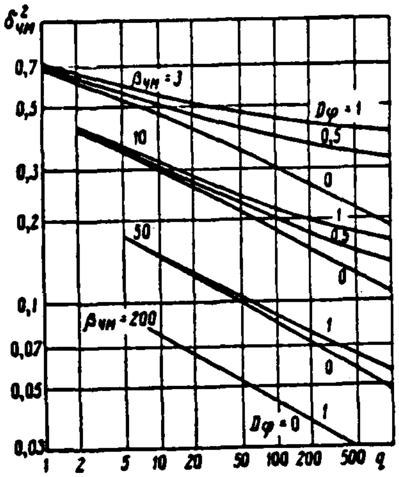


Fig. X.8.

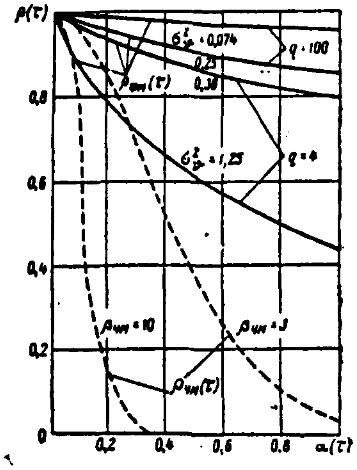


Fig. X.9.

Fig. X.8. Dependence of the square of the relative error of the filtration of communication/report on the signal-to-noise ratio with the frequency modulation.

Fig. X.9. Graphs of the correlation coefficients.

Page 301.

Taking into account relationship/ratio (X.55) Fig. X.9 gives the graphs of the coefficient of correlation $\rho_{ЧМ}(\tau)$ (broken lines) for two values $\beta_{ЧМ}=3$ and 10 and the coefficient of correlation $\rho_{ФМ}(\tau)$ (solid lines) for appropriate values $\sigma_\varphi^2$ with q=4 and 100. It follows from the graphs that the energy spectrum of the FM-signal is narrower than

the spectrum of the ChM-signal. Therefore phase modulation has an advantage over frequency modulation.

Frequency modulation in the presence of fadings. Let us assume that the useful radio signal takes the form

$$S(t, \lambda) = A(t)\cos(\omega_0 t + \psi(t));$$
$$\psi(t) = \varphi(t) + M_\psi \int_0^t \lambda(\tau)\,d\tau; \quad A(t) \geqslant 0; \qquad (X.58)$$

here A(t) and $\psi(\tau)$ - the random processes, the first of which considers the amplitude fading of radio signal, and the second contains information about the parameter λ(t) interesting to us.

Let both processes be Markovian and described by equations

$$\left.\begin{aligned} \dot{\psi} &= M_\psi \lambda + n_\varphi(t); \\ \dot{\lambda} &= -\alpha\lambda + n_\lambda(t); \\ \dot{A} &= -\gamma A + \frac{1}{4A} N_A + n_A(t), \end{aligned}\right\} \qquad (X.59)$$

where noises $n_\lambda(t)$ and $n_\varphi(t)$ make previous sense, and $n_A(t)$ — white noise with an one-way spectral density of $N_A$.

Let us note that the case of Rayleigh signal fading here is examined, since it follows from latter/last equation (X.59) that the one-dimensional probability density of process A(t) in the steady state coincides with Rayleigh's law:

$$p(A) = \frac{A}{\sigma^2}\exp\left(-\frac{A^2}{2\sigma^2}\right); \quad \sigma^2 = \frac{1}{4\gamma} N_A; \quad A \geqslant 0, \qquad (X.60)$$

With simplifications indicated earlier in the equation of optimum nonlinear filtration it is possible to reduce to the following form:

$$\left.\begin{aligned} \dot{\psi}^* &= M_\psi \lambda^* + \bar{K}^*_{\psi\psi} F_\psi; \quad \dot{\lambda}^* = -\alpha\lambda^* + \bar{K}^*_{\lambda\psi} F_\psi; \\ \dot{A}^* & \quad -\gamma A^* + \frac{1}{4A^*} N_A + \bar{K}^*_{AA} F_A, \end{aligned}\right\} \qquad (X.61)$$

where

$$\left.\begin{aligned} F_\psi &= -\frac{2A^*}{N_0}\xi(t)\sin(\omega_0 t + \psi^*); \\ F_A &= \frac{1}{N_0}[2\xi(t)\cos(\omega_0 t + \psi^*) - A^*]. \end{aligned}\right\} \qquad (X.62)$$

Page 302.

We will obtain as a result of solving the corresponding system of equations that for the errors the filtration remains valid of formulas (X.53), which it is necessary to supplement with the relationship/ratio

$$\bar{K}^*_{AA} = \frac{3}{2}\gamma N_0\left(\sqrt{1 + \frac{2N_A}{9\gamma^2 N_0}} - 1\right). \qquad (X.63)$$

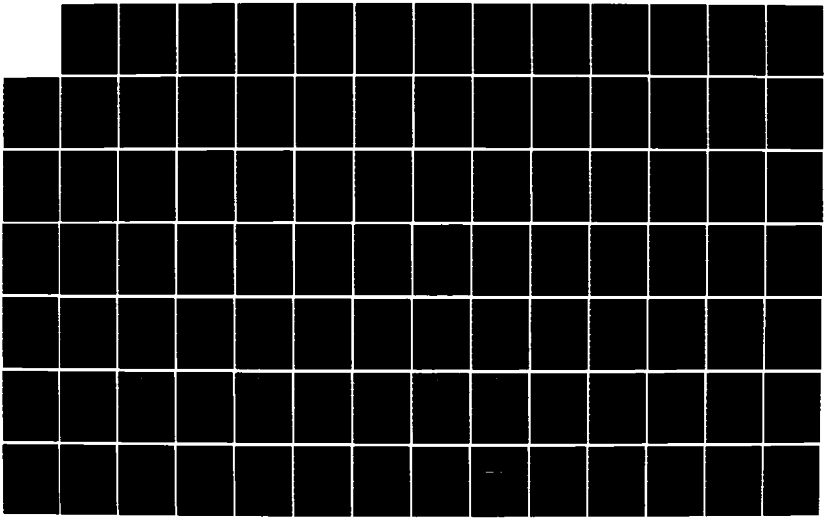
moreover in this case signal-to-noise ratio it is equal

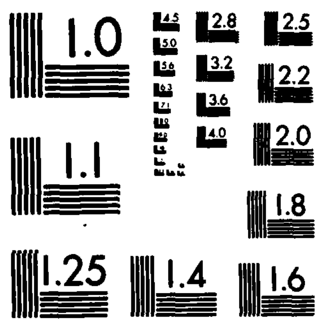
$$q - \frac{M[A^2]}{2\alpha N_0} = \frac{N_A}{4\alpha\gamma N_0}; \quad M[A^2] - 2\sigma^2 - \frac{1}{2\gamma} N_A. \qquad (X.64)$$

The structural scheme of the optimum filtering device/equipment

AD-A143 297 STATIC METHODS IN THE DESIGN OF NONLINEAR AUTOMATIC CONTROL SYSTEMS(U) FOREIGN TECHNOLOGY DIV WRIGHT-PATTERSON AFB OH N I ANDREYEV ET AL. 27 JUN 84 7/9

UNCLASSIFIED FTD-ID(RS)T-1734-83 F/G 12/1 NL






MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

is determined by equations (X.61), which taking into account expressions (X.62) can be recorded in the following form:

$$\left.\begin{aligned}
\dot{\psi}^* &= M_\psi \lambda^* - \frac{2\overline{K}^*_{\psi\psi}}{N_0} A^* \xi(t) \sin(\omega_0 t + \psi^*);\\
\dot{\lambda}^* &= -\alpha\lambda^* - \frac{2\overline{K}^*_{\lambda\psi}}{N_0} A^* \xi(t) \sin(\omega_0 t + \psi^*);\\
\dot{A}^* &= -\gamma' A^* + \frac{1}{4A^*} N_A + \frac{2\overline{K}^*_{AA}}{N_0} \xi(t) \cos(\omega_0 t + \psi^*);\\
\gamma' &= \frac{1}{2}\gamma\left(3\sqrt{1 + \frac{2N_A}{9\gamma^2 N_0}} - 1\right).
\end{aligned}\right\} \quad (X.65)$$

The possible version of the schematic of the optimum receiver, which simulates these equations, is depicted in Fig. X.10. In contrast to the diagram in Fig. X.7 this receiver has the special system of the automatic gain control, which works as follows. With the reduction of the amplitude of the useful signal adopted the amplification factor for the fundamental informational channel is reduced, i.e., the suppression of weak signal is realized.

The value of the square of the relative error of the filtration of the parameter is determined by expression (X.54), in which the signal-to-noise ratio is determined by formulas (X.64). Therefore for fading signal (X.58) in question remain valid graphs, given in Fig. X.8.

Page 303.

3. Synthesis of the optimum continuous communication systems for the two-component informational communications/reports.

Ordinary amplitude modulation. Let us consider all previous forms of modulation, when informational communication/report is two-component Markov process, i.e., informational communication/report is described by the equations

$$\left.\begin{aligned} \dot{\lambda} + \alpha\lambda &= -\beta x + n_{\lambda}(t); \\ \dot{x} &= -\beta x + n_{\lambda}(t) \end{aligned}\right\} \qquad (X.66)$$

instead of equation (X.8).

The spectrum of this communication/report with $\alpha=\beta$ satisfactorily reflects the character of the spectrum of real speech and, it, therefore, can serve as the model of speech. Useful radio signal with the amplitude modulation (AM) can be recorded in the form

$$\left.\begin{aligned} S(t, \lambda) &= [A_0 + M_A\lambda(t)]\cos[\omega_0 t + \psi(t)]; \\ \psi(t) &= \Phi(t) + \varphi(t), \end{aligned}\right\} \qquad (X.67)$$

where $A_0$ and $\omega_0$ - the a priori known values of amplitude and frequency:

$\lambda(t)$ - informational parameter;

$M_A$ — constant coefficient;

$\Phi(t)$ - the random phase, which is changed due to the Doppler effect;

$\varphi(t)$ - the random phase, which is changed due to the instability of the frequency of the master oscillator of transmitter.

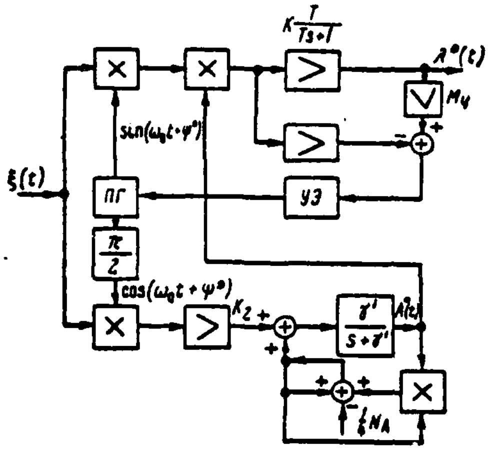


Fig. X.10. Version of the structural scheme of the optimum sensing transducer of the frequency modulated in the presence of amplitude fadings.

Page 304.

let us assume that the random phase $\psi(t)$ is described by the a priori stochastic equation

$$\dot{\psi} = \omega - \omega_0 + n_\varphi(t);\ \dot{\omega} = -\gamma(\omega - \omega_0) + n_\omega(t), \qquad (X.68)$$

where $n_\varphi(t)$, $n_\omega(t)$ — delta-correlated random processes with the zero average/mean values, which have with respect preset spectral intensities $N_\varphi$, $N_\omega$;

$\gamma$ - band of the Doppler spectrum at the level 0.5.

In connection with this case, in the gaussian approximation/approach of the equation of optimum nonlinear filtration for the steady state after some simplifications they take the form

$$\left.\begin{aligned}
\dot{\lambda}^* &= -\alpha\lambda^* - \beta x^* + K^*_{\lambda\lambda}F_\lambda + K^*_{\lambda\psi}F_\psi;\\
\dot{x}^* &= -\beta x^* + K^*_{\lambda\lambda}F_\lambda + K^*_{\lambda\psi}F_\psi;\\
\dot{\psi}^* &= (\omega^* - \omega_0) + K^*_{\psi\psi}F_\psi + K^*_{\lambda\psi}F_\lambda;\\
\dot{\omega}^* &= -\gamma(\omega^* - \omega_0) + K^*_{\psi\omega}F_\psi + K^*_{\lambda\omega}F_\lambda;
\end{aligned}\right\} \qquad (X.69)$$

$$\left.\begin{gathered}
\frac{1}{2}N_\lambda - 2\alpha\overline{K}^*_{\lambda\lambda} - 2\beta\overline{K}^*_{\lambda x} + (\overline{K}^*_{\lambda\lambda})^2\overline{F}_{\lambda\lambda} + (\overline{K}^*_{\lambda\psi})^2\overline{F}_{\psi\psi} = 0;\\
\frac{1}{2}N_\lambda - 2\beta\overline{K}^*_{xx} + (\overline{K}^*_{\lambda x})^2\overline{F}_{\lambda\lambda} + (\overline{K}^*_{x\psi})^2\overline{F}_{\psi\psi} = 0;\\
\frac{1}{2}N_\lambda - (\alpha+\beta)\overline{K}^*_{\lambda x} - \beta K^*_{xx} + \overline{K}^*_{\lambda\lambda}\overline{K}^*_{\lambda x}\overline{F}_{\lambda\lambda} +\\
+ \overline{K}^*_{\lambda\psi}\overline{K}^*_{x\psi}\overline{F}_{\psi\psi} = 0;\\
\frac{1}{2}N_\omega - 2\gamma\overline{K}^*_{\omega\omega} + (\overline{K}^*_{\omega\psi})^2\overline{F}_{\psi\psi} + (\overline{K}^*_{\lambda\omega})^2\overline{F}_{\lambda\lambda} = 0;\\
\frac{1}{2}N_\varphi + 2\overline{K}^*_{\psi\omega} + (\overline{K}^*_{\psi\psi})^2\overline{F}_{\psi\psi} + (\overline{K}^*_{\lambda\psi})^2\overline{F}_{\lambda\lambda} = 0;\\
\overline{K}^*_{\omega\omega} - \gamma\overline{K}^*_{\psi\omega} + \overline{K}^*_{\psi\psi}\overline{K}^*_{\psi\omega}\overline{F}_{\psi\psi} + \overline{K}_{\lambda\psi}\overline{K}_{\lambda\omega}\overline{F}_{\lambda\lambda} = 0;\\
\overline{K}^*_{\lambda\omega} - \beta\overline{K}^*_{x\psi} - \alpha\overline{K}^*_{\lambda\psi} + \overline{K}^*_{\lambda\lambda}\overline{K}^*_{\lambda\psi}\overline{F}_{\lambda\lambda} + \overline{K}^*_{\lambda\psi}\overline{K}^*_{\psi\psi}\overline{F}_{\psi\psi} = 0;\\
\overline{K}^*_{x\omega} - \beta\overline{K}^*_{x\psi} + \overline{K}^*_{\lambda x}\overline{K}^*_{\lambda\psi}\overline{F}_{\lambda\lambda} + \overline{K}^*_{x\psi}\overline{K}^*_{\psi\psi}\overline{F}_{\psi\psi} = 0;\\
-\gamma\overline{K}^*_{\lambda\omega} - \alpha\overline{K}^*_{\lambda\omega} - \beta\overline{K}^*_{x\omega} + \overline{K}_{\lambda\lambda}\overline{K}^*_{\lambda\omega}\overline{F}_{\lambda\lambda} + \overline{K}^*_{\lambda\psi}\times\\
\times\overline{K}^*_{\psi\omega}\overline{F}_{\psi\psi} = 0;\\
-\gamma\overline{K}^*_{x\omega} - \beta\overline{K}^*_{x\omega} + \overline{K}^*_{\lambda x}\overline{K}^*_{\lambda\omega}\overline{F}_{\lambda\lambda} +\\
+ \overline{K}^*_{x\psi}\overline{K}^*_{\psi\omega}\overline{F}_{\psi\psi} = 0.
\end{gathered}\right\} \qquad (X.70)$$

Page 305.

The solution of system (X.70) leads to the following results:

$$\left.\begin{aligned}
\overline{K}^{*}_{\lambda\lambda} &= \overline{K}^{*}_{\lambda x} = \frac{(\alpha+\beta)N_0}{M_A^2}\left(\sqrt{1+\frac{M_A^2 N_\lambda}{2(\alpha+\beta)^2 N_0}}-1\right);\\
\overline{K}^{*}_{xx} &= \frac{(\alpha+\beta)^2 N_0}{\beta M_A^2}\left(\sqrt{1+\frac{M_A^2 N_\lambda}{2(\alpha+\beta)^2 N_0}}-1\right);\\
\overline{K}^{*}_{\psi\psi} &= \frac{\gamma N_0}{A_0^2+\sigma_A^2}\left(\sqrt{1+L+2G}-1\right);\\
\overline{K}^{*}_{\omega\omega} &= \frac{\gamma^3 N_0}{A_0^2+\sigma_A^2}\left(1+G-\sqrt{1+L+2G}\right)\sqrt{1+L+2G};\\
\overline{K}^{*}_{\psi\omega} &= \frac{\gamma^2 N_0}{A_0^2+\sigma_A^2}\left(1+G-\sqrt{1+L+2G}\right);\\
\overline{K}^{*}_{\lambda\psi} &= \overline{K}^{*}_{x\psi} = \overline{K}^{*}_{\lambda\omega} = \overline{K}^{*}_{x\omega} = 0,
\end{aligned}\right\} \quad (X.71)$$

where $L = \frac{(A_0^2+\sigma_A^2)N_\varphi}{2\gamma^2 N_0}$; $G = \sqrt{\frac{(A_0^2+\sigma_A^2)(N_\omega+\gamma^2 N_\varphi)}{2\gamma^4 N_0}}$.

The freedom from interference of the reception of continuous signals we will characterize with the value of the relative error of the filtration of communication/report, which taking into account equality $N_\lambda = 8\alpha\sigma_\lambda^2$, valid with $\alpha=\beta$, is equal to

$$\delta_{AM}^2 = \frac{\overline{K}^{*}_{\lambda\lambda}}{\sigma_\lambda^2} = \frac{2\alpha N_0}{\sigma_A^2}\left(\sqrt{1+\frac{\sigma_A^2}{\alpha N_0}}-1\right). \qquad (X.72)$$

Through formula (X.11) we find the average/mean power of useful signal (X.67):

$$P = \frac{1}{T}\int_0^T S^2(t)\,dt = \frac{1}{2}(A_0^2+\sigma_A^2) = \frac{1}{2}\sigma_A^2\left(\frac{1+m^2}{m^2}\right),\ \omega_0 T \gg 1,$$

where $m = \frac{\sigma_A}{A_0}$ — coefficient of amplitude modulation.

Signal-to-noise ratio at the input of receiver is equal

$$q = \frac{P}{\alpha N_0} = \frac{\sigma_A^2}{2\alpha N_0} \cdot \frac{1+m^2}{m^2}.$$

Page 306.

From expression (X.72) we obtain final formula for the square of the relative error of the filtration of the communication/report:

$$\delta_{AM}^2 = \frac{1}{q\frac{m^2}{1+m^2}}\left(\sqrt{1+2q\frac{m^2}{1+m^2}}-1\right), (\alpha=\beta). \quad \text{(X.73)}$$

In the approximation/approach examined, as in the one-dimensional case, the relative error of the filtration of the amplitude-modulated signals does not depend on the value of the phase fluctuations of signal.

After substituting in equations (X.69) the obtained average/mean values of cumulants (X.71) and expression

$$F_\lambda = \frac{\partial F}{\partial \lambda^*} \quad \frac{M_A}{N_0}[2\xi(t)\cos(\omega_0 t + \psi^*) - (A_0 + M_A\lambda^*)];$$

$$F_\psi = \frac{\partial F}{\partial \psi^*} = -\frac{2}{N_0}(A_0 + M_A\lambda^*)\xi(t)\sin(\omega_0 t + \psi^*),$$

we will obtain

$$\left.\begin{aligned}\lambda^* &= \frac{K_\lambda}{T_x s+1}\frac{T_\lambda s}{T_\lambda s+1}\left[2\xi(t)\cos(\omega_0 t+\psi^*)-(A_0+M_A\lambda^*)\right];\\ \dot{\psi}^* &= -S_y\left(K_\psi+\frac{K_\omega}{T_\omega s+1}\right)\xi(t)(A_0+M_A\lambda^*)\sin(\omega_0 t+\psi^*),\end{aligned}\right\}\quad (X.74)$$

where

$$K_\psi=\frac{2\bar{K}^*_{\psi\psi}}{S_y N_0};\; K_\omega=\frac{2\bar{K}^*_{\psi\omega}}{S_y\gamma N_0};\; K_\lambda=\frac{M_A\bar{K}^*_{\lambda\lambda}}{\beta N_0};\; T_x=\beta^{-1};$$
$$=\frac{1}{\alpha};\; T_\omega=\frac{1}{\gamma};$$

$S_y$ – constant coefficient.

Equations (X.74) are simulated by the optimum receiver, whose structural scheme is given in Fig. X.11. receiver realizes quasi-coherent reception of signals. It has the fundamental, informational channel, at the output of which is obtained estimated value λ (t), and the phase automatic frequency control, which develops reference signal.

Informational channel consists of multiplier (synchronous detector) and optimum linear filter, formed series-connected and those included by negative feedback by amplifier and filters of upper and lower frequencies. The amplitude-frequency characteristic of optimum linear filter is determined by the spectrum of the transmitted communication/report and by the signal-to-noise ratio at the input of the receiver.

Page 307.

It is evident from a comparison of Fig. X.1 and Fig. X.11 that the structural schemes of optimum receivers for the one-component and two-component Markov communications/reports are characterized by only the type of linear filters. In the first case the linear filter, which forms informational communication/report from the white noise, is determined by linear differential first-order (X.8) equation, and secondly - by system (X.66) of two linear differential first-order equations.

This law has general character, i.e., it relates to all forms examined below of modulation, and specially it is not specified subsequently.

Amplitude modulation in the presence of fadings. The fading amplitude-modulated radio signal is represented in the form

$$S(t, \lambda) = E(t)[A_0 + M_A\lambda(t)]\cos(\omega_0 t + \psi), \qquad \text{(X.75)}$$

where E(t) - the random process, which considers fading radio signal, and the remaining parameters make previous sense. Let the process E(t) be Markovian. Then for signal (X.75) it is possible to write the following system of a priori stochastic equations:

$$\left.\begin{array}{c} \dot{\lambda} = -\alpha\lambda - \beta x + n_\lambda(t);\ \dot{x} = -\beta x + n_\lambda(t); \\ \dot{E} = -\mu E + \frac{1}{4E} N_E + n_E(t); \\ \dot{\psi} = (\omega - \omega_0) + n_\varphi(t);\ \dot{\omega} = -\gamma(\omega - \omega_0) + n_\omega(t). \end{array}\right\} \qquad \text{(X.76)}$$

Here noises $n_\lambda(t)$, $n_\varphi(t)$ and $n_\omega(t)$ make previous sense, and $n_E(t)$ — white noise with an one-way spectral density of $N_E$.

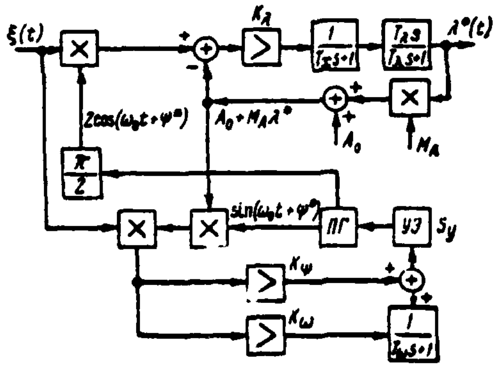



Fig. X.11. Structural scheme of the optimum receiver of the amplitude-modulated radio signals.

Page 308.

The simplified equations of optimum nonlinear filtration it is possible to reduce to the following form:

$$\left.\begin{aligned} \dot{\lambda}^* &= -\alpha\lambda^* - \beta\varkappa^* + \overline{K}^*_{\lambda\lambda}F_\lambda; \\ \dot{\varkappa} &= -\beta\varkappa^* + \overline{K}^*_{\lambda\lambda}F_\lambda; \end{aligned}\right\} \qquad (X.77)$$

$$\left.\begin{aligned} \dot{E}^* &= -\mu E^* + \frac{1}{4E^*}N_E + \overline{K}^*_{EE}F_E; \\ \dot{\psi}^* &= (\omega^* - \omega_0) + \overline{K}^*_{\psi\psi}F_\psi; \\ \dot{\omega}^* &= -\gamma(\omega^* - \omega_0) + \overline{K}^*_{\psi\omega}F_\psi; \end{aligned}\right\} \qquad (X.77)$$

$$\left.\begin{aligned} F_\lambda &= \frac{M_A}{N_0}[2\xi(t)E^*\cos(\omega_0 t + \psi^*) - (A_0 + M_A\lambda^*)(E^*)^2]; \\ F_E &= \frac{1}{N_0}[2\xi(t)(A_0 + M_A\lambda^*)\cos(\omega_0 t + \psi^*) - \\ &\quad -(A_0 + M_A\lambda^*)^2 E^*]; \\ F_\psi &= -\frac{2}{N_0}\xi(t)E^*(A_0 + M_A\lambda^*)\sin(\omega_0 t + \psi^*). \end{aligned}\right\} \qquad (X.78)$$

where

As a result of solving the corresponding system of equations we have

$$\left.\begin{aligned}
\overline{K}^{*}_{\lambda\lambda} &= \frac{(\alpha+\beta)N_0}{M_A^2 M[E^2]}\left(\sqrt{1+\frac{M_A^2 M[E^2]N_\lambda}{2(\alpha+\beta)^2 N_0}}-1\right);\\
\overline{K}^{*}_{EE} &= \frac{3\mu N_0}{2(A_0^2+\sigma_A^2)}\left(\sqrt{1+\frac{2(A_0^2+\sigma_A^2)N_E}{9\mu^2 N_0}}-1\right);\\
\overline{K}^{*}_{\psi\psi} &= \frac{\gamma N_0}{(A_0^2+\sigma_A^2)M[E^2]}\left(\sqrt{1+L+2G}-1\right);\\
\overline{K}^{*}_{\psi\omega} &= \frac{\gamma^2 N_0}{(A_0^2+\sigma_A^2)M[E^2]}\left(1+G-\sqrt{1+L+2G}\right),
\end{aligned}\right\} \qquad (X.79)$$

where

$$L=\frac{(A_0^2+\sigma_A^2)M[E^2]}{2\gamma^2 N_0};\quad G=\sqrt{\frac{(A_0^2+\sigma_A^2)M[E^2](N_\omega+\gamma^2 N_\varphi)}{\gamma^4 N_0}}.$$

For the relative error of the filtration of voice communication with the Rayleigh fadings it is possible to write

$$\delta_{AM}^2=\frac{1}{q\dfrac{m^2}{1+m^2}}\left(\sqrt{1+2q\frac{m^2}{1+m^2}}-1\right);$$

$$q=\frac{P}{\beta N_0}=\frac{\sigma_A^2}{2\beta N_0}M[E^2]. \qquad (X.80)$$

Page 309.

The structural scheme of the optimum sensing transducer of the fading amplitude-modulated is determined by equations (X.77), which taking into account expressions (X.78) can be recorded in the following form:

$$\left.\begin{aligned}
\lambda^* &= \frac{K_\lambda}{T_x s+1}\cdot\frac{T_\lambda s}{T_\lambda s+1}\,[2E^*\xi(t)\cos(\omega_0 t+\psi^*) -\\
&\quad -(A_0+M_A\lambda^*)(E^*)^2];\\
E^* &= \frac{1}{T_E s+1}\Big\{K_E\,[2(A_0+M_A\lambda^*\xi(t)\cos(\omega_0 t+\\
&\quad +\psi^*)-(A_0+M_A\lambda^*)^2E^*]+\frac{1}{2}\,\frac{M[E^2]}{E^*}\Big\};\\
\dot{\psi}^* &= -S_y\left(K_\psi+\frac{K_\omega}{T_\omega s+1}\right)[(A_0+M_A\lambda^*)E^*\xi(t)\times\\
&\quad\times\sin(\omega_0 t+\psi^*)].
\end{aligned}\right\}\quad (X.81)$$

Here

$$K_\lambda=\frac{M_A\overline{K}^*_{\lambda\lambda}}{\beta N_0};\quad K_E=\frac{K^*_{EE}}{\mu N_0};$$
$$K_\psi=\frac{2\overline{K}^*_{\psi\psi}}{S_yN_0};\quad K_\omega=\frac{2\overline{K}^*_{\psi\omega}}{S_y\mu N_0};$$
$$T_x=\beta^{-1};\ T_\lambda=\alpha^{-1};\ T_E=\mu^{-1};\ T_\omega=\gamma^{-1}.$$

The possible version of the structural scheme of the optimum receiver, which simulates these equations, is depicted in Fig. X.12. In contrast to the diagram in Fig. X.11 this receiver has a channel of the optimum extraction of the multiplicative interference E (t), which then is used for eliminating the effect of fadings.

Two-band modulation without the carrier. Let us record two-band signal without the carrier in the form

$$S(t,\lambda)=M_A\lambda(t)\cos[\omega_0 t+\psi(t)].\qquad (X.82)$$

The average/mean power of this signal is equal to

$$P=\frac{1}{T}\int_0^T S^2(t,\lambda)\,dt=\frac{1}{2}\sigma_A^2.$$

From a comparison of the recording of useful signals (X.67) and (X.82), it directly follows that for the signal in question, all the expression obtained above expressions are valid, and in them we must only assume that $A_0=0$.

Page 310.

For the square of the relative error of the filtration of voice communication with two-band modulation without the carrier we will obtain the formula

$$\delta^2_{ДМ}=\frac{1}{q}(\sqrt{1+2q}-1),\quad q=\frac{\sigma_A^2}{2\beta N_0}. \qquad (X.83)$$

In connection with signal (X.82) of equation (X.74) they take the form

$$\left.\begin{aligned}\lambda^*&=\frac{K_\lambda}{T_s s+1}\,\frac{T_\lambda s}{T_\lambda s+1}\,[2\xi(t)\cos(\omega_0 t+\psi^*)-M_A\lambda^*];\\ \psi^*&=-S_y\left(K_\psi+\frac{K_\omega}{T_\omega s+1}\right)\xi(t)\,M_A\lambda^*\sin(\omega_0 t+\psi^*).\end{aligned}\right\} \qquad (X.84)$$

Structural diagram of the receiver, constructed in accordance with equations (X.84), is depicted in Fig. X.13.

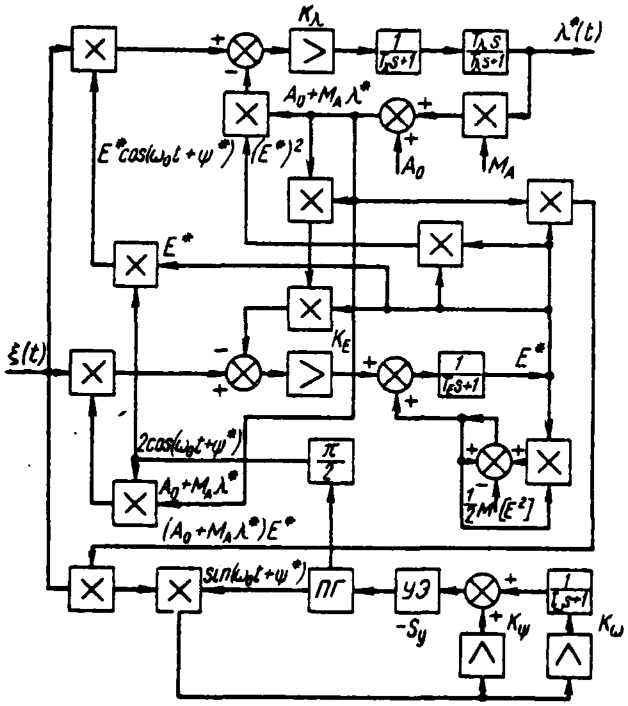



Fig. X.12. Structural scheme of the optimum receiver of amplitude-modulated radio signals in the presence of fadings.

Page 311.

Single-band modulation with the pilot signal. Useful radio signal in single-band modulation and presence of the pilot signal, which contains information about the phase, can be represented in the following form:

$$S(t, \lambda) = \frac{1}{\sqrt{2}} M_A \lambda(t) \cos[\omega_0 t + \psi(t)] + \frac{1}{\sqrt{2}} M_A \hat{\lambda}(t) \sin[\omega_0 t + \psi(t)] + C_0 \cos[\omega_0 t + \psi(t)]. \quad (X.85)$$

where $\hat{\lambda}(t)$ - the transformation of Gilbert/Hilbert from the communication/report $\lambda(t)$;

$C_0$ - known constant value of the amplitude of pilot signal.

As earlier, we will consider that the input of the channel of the formation of supporting/reference oscillation enters the sum of pilot signal and white noise:

$$\eta(t) \approx C_0 \cos[\omega_0 t + \psi(t)] + n(t).$$

Let us assume that the random parameters of single-band signal are described by the a priori stochastic equations

$$\left.\begin{aligned} \dot{\lambda} &= -\alpha\lambda - \beta x + n_\lambda(t);\ \dot{x} = -\beta x + n_\lambda(t), \\ \dot{\hat{\lambda}} &= -\alpha\hat{\lambda} - \beta\hat{x} + \hat{n}_\lambda(t);\ \dot{\hat{x}} = -\beta\hat{x} + \hat{n}_\lambda(t); \\ \dot{\psi} &= \omega - \omega_0 + n_\psi(t);\ \dot{\omega} = -\gamma(\omega - \omega_0) + n_\omega(t), \end{aligned}\right\} \quad (X.86)$$

where $\hat{x}(t)$ - the transformation of Gilbert/Hilbert from process of x(t);

$\hat{n}_\lambda(t)$ — transformation of Gilbert/Hilbert from white noise $n_\lambda(t)$.

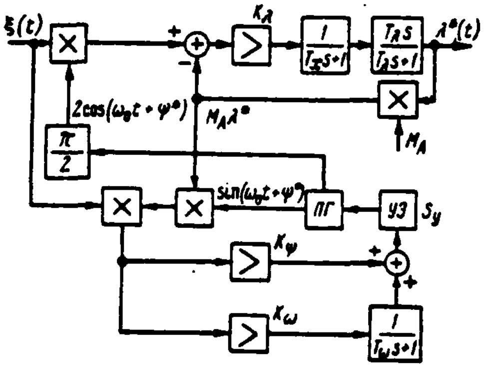



Fig. X.13. Structural scheme of the optimum receiver of two-band radio signal with the amplitude modulation.

Page 312.

If we use the previous procedure of simplification in the expressions for the errors of filtration (namely, to substitute in the corresponding equations the time-averaged expressions of functions $F_\mu$, and then to switch over to steady state), then let us arrive at the following system of equations of the optimum nonlinear filtration:

$$\left.\begin{aligned}
\dot{\lambda}^* &= -\alpha\lambda^* - \beta x^* + \overline{K}^*_{\lambda\lambda}F_\lambda;\\
\dot{x}^* &= -\beta x^* + \overline{K}^*_{\lambda\lambda}F_\lambda;\\
\dot{\hat{\lambda}}^* &= -\alpha\hat{\lambda}^* - \beta\hat{x}^* + \overline{K}^*_{\hat{\lambda}\hat{\lambda}}F_{\hat{\lambda}};\\
\dot{\hat{x}}^* &= -\beta\hat{x}^* + \overline{K}^*_{\hat{\lambda}\hat{\lambda}}F_{\hat{\lambda}};\\
\dot{\psi}^* &= (\omega^* - \omega_0) + \overline{K}^*_{\psi\psi}\Phi_\psi;\\
\dot{\omega}^* &= -\gamma(\omega^* - \omega_0) + \overline{K}^*_{\psi\omega}\Phi_\psi.
\end{aligned}\right\} \quad (X.87)$$

Here

$$
\left.\begin{aligned}
F &= \frac{1}{N_0}\left[2\xi(t)\,S(t,\lambda^*) - S^2(t,\lambda^*)\right];\\
\Phi &= \frac{2}{N_0} C_0 \eta(t)\cos(\omega_0 t + \psi^*);\\
F_\lambda &= \frac{M_A}{N_0}\left[\sqrt{2}\,\xi(t)\cos(\omega_0 t + \psi^*) - \frac{1}{2} M_A \lambda^*\right];\\
F_{\hat\lambda} &= \frac{M_A}{N_0}\left[\sqrt{2}\,\xi(t)\sin(\omega_0 t + \psi^*) - \frac{1}{2} M_A \hat\lambda^*\right];\\
\Phi_\psi &= -\frac{2}{N_0} C_0 \eta(t)\sin(\omega_0 t + \psi^*);\\
\overline{K}^*_{\lambda\lambda} = \overline{K}^*_{\hat\lambda\hat\lambda} &= \frac{2(\alpha+\beta)N_0}{M_A^2}\left(\sqrt{1 + \frac{M_A^2 N_\lambda}{4(\alpha+\beta)^2 N_0}} - 1\right);\\
\overline{K}^*_{\psi\psi} &= \frac{\gamma N_0}{C_0^2}\left(\sqrt{1+L+2G} - 1\right);\\
\overline{K}^*_{\psi\omega} &= \frac{\gamma^2 N_0}{C_0^2}\left(1 + G - \sqrt{1+L+2G}\right),
\end{aligned}\right\} \quad \text{(X.88)}
$$

where

$$L = \frac{C_0^2 N_\varphi}{2\gamma^2 N_0}; \quad G = \sqrt{\frac{C_0^2 (N_\omega + \gamma^2 N_\varphi)}{2\gamma^4 N_0}}.$$

The average/mean power of radio signal (X.85) is equal to

$$P = \frac{1}{T}\int_0^T S^2(t,\lambda)\,dt = \frac{1}{2}\left(M_A^2\sigma_\lambda^2 + C_0^2\right) = \frac{1}{2}\left(\sigma_A^2 + C_0^2\right).$$

Page 313.

Signal-to-noise ratio at the input can be represented in the form

$$q = \frac{P}{\beta N_0} = \frac{\sigma_A^2}{2\beta N_0}\left(\frac{1+m_0^2}{m_0^2}\right), \qquad \text{(X.89)}$$

where $m_0^2 = \sigma_A^2 / C_0^2$— separation factor of power between the

informational radio signal and the pilot signal.

After substituting into the formula for $\bar{K}^{*}_{\lambda\lambda}$ expression $\sigma_{\lambda}^{2}$ and $N_{\lambda}=8\beta\sigma_{\lambda}^{2}$ and after introducing signal-to-noise ratio (X.89), we can write

$$\bar{K}^{*}_{\lambda\lambda}=\frac{2\sigma_{\lambda}^{2}}{q\dfrac{m_0^2}{1+m_0^2}}\left(\sqrt{1+q\frac{m_0^2}{1+m_0^2}}-1\right). \qquad (X.90)$$

Equations (X.86) of optimum nonlinear filtration after the substitution in them of corresponding relationships/ratios (X.88) take the form

$$\left.\begin{aligned}
\lambda^{*}&=\frac{K_{\lambda}}{T_{x}s+1}\cdot\frac{T_{\lambda}s}{T_{\lambda}s+1}\left[\sqrt{2}\,\xi(t)\cos(\omega_0 t+\psi^{*})-\frac{1}{2}M_A\lambda^{*}\right];\\
\hat{\lambda}^{*}&=\frac{K_{\lambda}}{T_{x}s+1}\cdot\frac{T_{\lambda}s}{T_{\lambda}s+1}\left[\sqrt{2}\,\xi(t)\sin(\omega_0 t+\psi^{*})-\frac{1}{2}M_A\hat{\lambda}^{*}\right];\\
\psi^{*}&=-S_{y}\left(K_{\psi}+\frac{K_{\omega}}{T_{\omega}s+1}\right)\eta(t)C_0\sin(\omega_0 t+\psi^{*}),
\end{aligned}\right\} \qquad (X.91)$$

where

$$K_{\psi}=\frac{2\bar{K}^{*}_{\psi\psi}}{S_y N_0};\ K_{\omega}=\frac{2\bar{K}^{*}_{\psi\omega}}{S_y\gamma N_0};\ K_{\lambda}=\frac{K^{*}_{\lambda\lambda}}{\beta N_0};\ T_x=\beta^{-1};$$
$$T_{\lambda}=\alpha^{-1};\ T_{\omega}=\gamma^{-1}.$$

Fig. X.14 gives the structural scheme of the optimum receiver, which simulates equations (X.91). Besides the channel of the synchronizations, the receiver has two informational channels, at outputs of which are selected estimated values $\lambda^{*}(t)$ and $\hat{\lambda}^{*}(t)$.

The computation of the resulting error of filtration after association of both channels (see Fig. X.3) in this case is made more complicatedly than in the previously one-dimensional case examined, and here it is not given.

Fig. X.15 presents the results of the calculations of the square of the relative error of the filtration of informational communication/report for the forms of amplitude modulation examined.

Page 314.

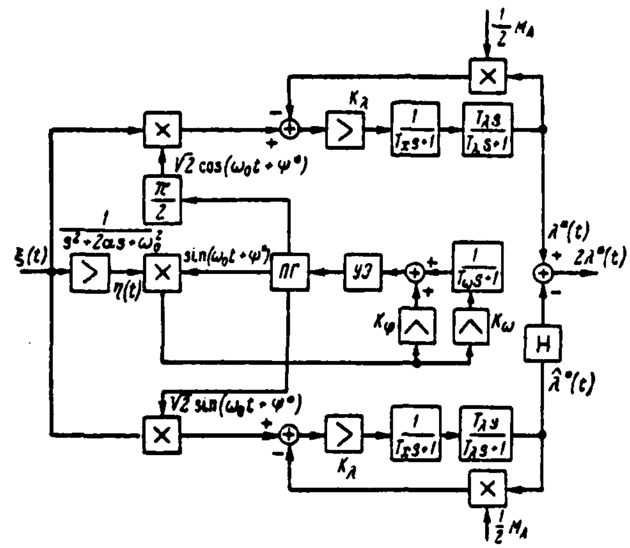


Fig. X.14. Structural scheme of the optimum receiver of single-band radio signals with the amplitude modulation.

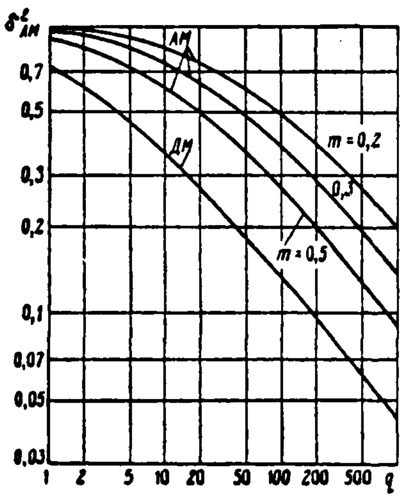


Fig. X.15. Dependence of the square of the relative error of the filtration of communication/report on the signal-to-noise ratio with the different types of amplitude modulation.

Page 315.

Phase modulation. With phase modulation useful radio signal takes the form

$$S(t, \lambda) = A_0 \cos[\omega_0 t + \psi(t)];$$
$$\psi(t) = \vartheta(t) + \varphi(t) = M_\varphi \lambda(t) + \varphi(t), \qquad (X.92)$$

where $A_0$ and $\omega_0$ - a priori known values of amplitude and frequency;

$M_\varphi = \sigma_\vartheta / \sigma_x$ — constant coefficient (mutual conductance of phase modulator).

The behavior of the random parameters of radio signal (X.92) is described by the system of the a priori stochastic equations

$$\left.\begin{aligned} \dot\psi &= M_\varphi \dot\lambda + \dot\varphi(t); \\ \dot\lambda &= -\alpha\lambda - \beta x + n_\lambda(t); \\ \dot x &= -\beta x + n_\lambda(t); \\ \dot\varphi &= n_\varphi(t). \end{aligned}\right\} \qquad (X.93)$$

Correlation and mutual-correlation functions are such in this case:

$$\left.\begin{aligned} &M[n_\lambda(t_1) n_\lambda(t_2)] = \frac{1}{2} N_\lambda \delta(\tau); M[(n_\varphi(t_1) n_\varphi(t_2)] = \frac{1}{2} N_\varphi \delta(\tau); \\ &M[n_\psi(t_1) n_\psi(t_2)] = \frac{1}{2} M_\varphi^2 N_\lambda \delta(\tau) + \frac{1}{2} N_\varphi \delta(\tau); \\ &M[n_\lambda(t_1) n_\psi(t_2)] = \frac{1}{2} M_\varphi N_\lambda \delta(\tau); \; n_\psi(t) = M_\varphi n_\lambda(t) + n_\varphi(t). \end{aligned}\right| \qquad (X.94)$$

The equations of optimum nonlinear filtration take the form

$$\left.\begin{aligned}&2\alpha M_{\varphi}\overline{K}^{*}_{\psi\lambda}+2\beta M_{\varphi}\overline{K}^{*}_{\psi x}-\frac{1}{2}M^{2}_{\varphi}N_{\lambda}-\frac{1}{2}N_{\varphi}+(\overline{K}^{*}_{\psi\psi})^{2}\overline{F}_{\psi\psi}=0;\\&2\alpha\overline{K}^{*}_{\lambda\lambda}+2\beta\overline{K}^{*}_{\lambda x}-\frac{1}{2}N_{\lambda}-(\overline{K}^{*}_{\psi\lambda})^{2}\overline{F}_{\psi\psi}=0;\\&2\beta\overline{K}^{*}_{xx}-\frac{1}{2}N_{\lambda}-(\overline{K}^{*}_{\psi x})^{2}\overline{F}_{\psi\psi}=0;\\&\alpha M_{\varphi}\overline{K}^{*}_{\lambda\lambda}+\beta M_{\varphi}\overline{K}^{*}_{\lambda x}+\alpha\overline{K}^{*}_{\psi\lambda}+\beta\overline{K}^{*}_{\psi x}-\frac{1}{2}M_{\varphi}N_{\lambda}-\\&\quad-\overline{K}^{*}_{\psi\psi}\overline{K}^{*}_{\psi\lambda}\overline{F}_{\psi\psi}=0;\\&\alpha M_{\varphi}\overline{K}^{*}_{\lambda x}+\beta M_{\varphi}\overline{K}^{*}_{xx}+\beta\overline{K}^{*}_{\psi x}-\frac{1}{2}M_{\varphi}N_{\lambda}-\overline{K}^{*}_{\psi\psi}\overline{K}^{*}_{\psi x}\overline{F}_{\psi\psi}=0;\\&\alpha\overline{K}^{*}_{\lambda x}+\beta\overline{K}^{*}_{xx}+\beta\overline{K}^{*}_{\lambda x}-\frac{1}{2}N_{\lambda}+\overline{K}^{*}_{\psi\lambda}\overline{K}^{*}_{\psi x}\overline{F}_{\psi\psi}=0;\end{aligned}\right\}.\quad (X.95)$$

$$\left.\begin{aligned}\dot{\psi}^{*}&=M_{\varphi}\lambda^{*}+K^{*}_{\psi\psi}F_{\psi};\\\dot{\lambda}^{*}&=-\alpha\lambda^{*}-\beta x^{*}+\overline{K}^{*}_{\psi\lambda}\overline{F}_{\psi};\\\dot{x}^{*}&=-\beta x^{*}+\overline{K}^{*}_{\psi x}\overline{F}_{\psi}.\end{aligned}\right\}\quad (X.96)$$

Page 316.

For signal (X.92) we have

$$F_{\psi}=-\frac{2}{N_{0}}\xi(t)A_{0}\sin(\omega_{0}t+\psi^{*});\ \overline{F}_{\psi\psi}=-\frac{A_{0}^{2}}{N_{0}}.\quad (X.97)$$

Let us introduce into the examination signal-to-noise ratio $q=A_0^2/2\alpha N_0$. Then taking into account expressions (X.92) (X.92) and (X.97) with $\beta=\alpha$ from system (X.95) we obtain

$$\left.\begin{aligned}
&2\sigma_\Phi \frac{\overline{K}^*_{\psi\lambda}}{\sigma_\lambda} + 2\sigma_\Phi \frac{\overline{K}^*_{\psi x}}{\sigma_\lambda} - 4\sigma_\Phi^2 - D_\Phi + 2q\left(\overline{K}^*_{\psi\psi}\right)^2 = 0;\\
&\overline{K}^*_{\lambda\lambda} + \overline{K}^*_{\lambda x} - 2\sigma_\lambda^2 + q\left(\overline{K}^*_{\psi\lambda}\right)^2 = 0;\\
&\overline{K}^*_{xx} - 2\sigma_\lambda^2 + q\left(\overline{K}^*_{\psi x}\right)^2 = 0;\\
&\sigma_\Phi \frac{\overline{K}^*_{\lambda\lambda}}{\sigma_\lambda} + \sigma_\Phi \frac{\overline{K}^*_{\lambda x}}{\sigma_\lambda} + \overline{K}^*_{\psi\lambda} + \overline{K}^*_{\psi x} - 4\sigma_\Phi\sigma_\lambda + 2q\overline{K}^*_{\psi\psi}\overline{K}^*_{\psi\lambda} = 0;\\
&\sigma_\Phi \frac{\overline{K}^*_{\lambda x}}{\sigma_\lambda} + \sigma_\Phi \frac{\overline{K}^*_{xx}}{\sigma_\lambda} + \overline{K}^*_{\psi x} - 4\sigma_\Phi\sigma_\lambda + 2q\overline{K}^*_{\psi\psi}\overline{K}^*_{\psi x} = 0;\\
&2\overline{K}^*_{\lambda x} + \overline{K}^*_{xx} - 4\sigma_\lambda^2 + 2q\overline{K}^*_{\psi\lambda}\overline{K}^*_{\psi x} = 0,
\end{aligned}\right\} \quad (X.98)$$

where, as earlier, $D_\Phi = N_\Phi/2a$.

For the transition/junction to the dimensionless quantities let us divide the fourth and fifth equations on $\sigma_\lambda$, and the second, the third and the sixth - on $\sigma_\lambda^2$. Then system of equations (X.98) will take the form

$$\left.\begin{aligned}
&2\sigma_\Phi \frac{\overline{K}^*_{\psi\lambda}}{\sigma_\lambda} + 2\sigma_\Phi \frac{\overline{K}^*_{\psi x}}{\sigma_\lambda} - 4\sigma_\Phi^2 - D_\Phi + 2q\left(\overline{K}^*_{\psi\psi}\right)^2 = 0;\\
&\delta^2_{\Phi M} + \frac{\overline{K}^*_{\lambda x}}{\sigma_\lambda^2} - 2 + q\left(\frac{\overline{K}^*_{\psi\lambda}}{\sigma_\lambda}\right)^2 = 0;\\
&\frac{\overline{K}^*_{xx}}{\sigma_\lambda^2} - 2 + q\left(\frac{\overline{K}^*_{\psi x}}{\sigma_\lambda}\right)^2 = 0;\\
&\sigma_\Phi\delta^2_{\Phi M} + \sigma_\Phi \frac{\overline{K}^*_{\lambda x}}{\sigma_\lambda^2} + \frac{\overline{K}^*_{\psi\lambda}}{\sigma_\lambda} + \frac{\overline{K}^*_{\psi x}}{\sigma_\lambda} - 4\sigma_\Phi + 2q\overline{K}^*_{\psi\psi} \frac{\overline{K}^*_{\psi\lambda}}{\sigma_\lambda} = 0;\\
&\sigma_\Phi \frac{\overline{K}^*_{\lambda x}}{\sigma_\lambda^2} + \sigma_\Phi \frac{\overline{K}^*_{xx}}{\sigma_\lambda^2} + \frac{\overline{K}^*_{\psi x}}{\sigma_\lambda} - 4\sigma_\Phi + 2q\overline{K}^*_{\psi\psi} \frac{\overline{K}^*_{\psi x}}{\sigma_\lambda} = 0;\\
&2\frac{\overline{K}^*_{\lambda x}}{\sigma^2} + \frac{\overline{K}^*_{xx}}{\sigma_\lambda^2} - 4 + 2q\frac{\overline{K}^*_{\psi\lambda}}{\sigma_\lambda}\frac{\overline{K}^*_{\psi x}}{\sigma_\lambda} = 0,
\end{aligned}\right\} \quad (X.99)$$

where $\delta^2_{\Phi M} = \hat{\overline{K}}^*_{\lambda\lambda}/\sigma_\lambda^2$.

Page 317.

When $D_{*}=0$ system (X.99) can be solved analytically. Expressions for $\delta^2_{\Phi M}$, $\bar{K}^{*}_{\lambda x}/\sigma^2_{\lambda}$, $\bar{K}^{*}_{xx}/\sigma^2_{\lambda}$, $\bar{K}^{*}_{\psi\lambda}/\sigma_{\lambda}$, $\bar{K}^{*}_{\psi x}/\sigma_{\lambda}$ and $\bar{K}^{*}_{\psi\psi}$ in this special case take the form

$$\sigma^2_{\Phi M}=\frac{\bar{K}^{*}_{\lambda\lambda}}{\sigma^2_{\lambda}}=\frac{\bar{K}^{*}_{\lambda x}}{\sigma^2_{\lambda}}=\frac{1}{\sigma^2_{\psi}q}\left(\sqrt{1+2\sigma^2_{\psi}q}-1\right);$$

$$\frac{\bar{K}^{*}_{xx}}{\sigma^2_{\lambda}}\quad\frac{2}{\sigma^2_{\psi}q}\left(\sqrt{1+2\sigma^2_{\psi}q}-1\right);$$

$$\frac{\bar{K}^{*}_{\psi\lambda}}{\sigma_{\lambda}}=\frac{\bar{K}^{*}_{\psi x}}{\sigma_{\lambda}}=\frac{1}{\sigma_{\psi}q}\left(\sqrt{1+2\sigma^2_{\psi}q}-1\right);$$

$$\bar{K}^{*}_{\psi\psi}=\frac{1}{q}\left(\sqrt{1+2\sigma^2_{\psi}q}-1\right).$$

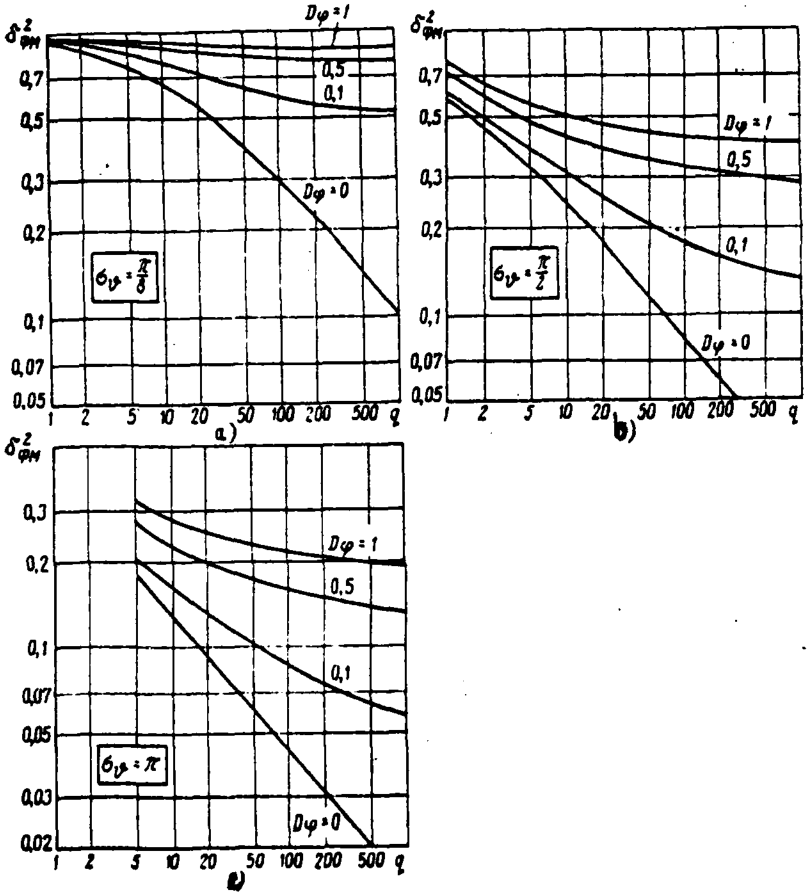



Fig. X.16. Dependence of the square of the relative error of the filtration of communication/report on the signal-to-noise ratio with phase modulation.

Page 318.

The results of solving the system of equations (X.99) numerical methods on TsVM for a series/row of values $\sigma_{\theta}$ and $D_{\varphi}$ depending on q are given in Fig. X.16-X.18 (when $K^{*}_{\psi\psi} \leqslant 2$ rad²).

Fig. X.16 depicts the graph/diagrams of the dependence of the error of filtration $\delta^{2}_{\Phi M}$ on the signal-to-noise ratio q. From the graphs it is evident that for the given values of q and $\sigma_{\theta}$ the error of filtration is minimum when $D_{\varphi} \doteq 0$ and it increases with increase/growth $D_{\varphi}$. The graphs, which characterize the dependence of the single cumulants, entering system (X.99), from the signal-to-noise ratio for several values $\sigma_{\theta}$ and $D_{\varphi}$, are represented in Fig. X.17, X.18.

The equations, which determine the structure of the optimum receiver of FM radio signals, can be obtained from the system of equations (X.96), after substituting value $F_{\psi}$ from expressions (X.97).

As a result of simple transformations we will have

$$\left.\begin{aligned}
\dot{\psi}^* &= S_y\left\{K\left[-\frac{1}{Ts+1}\left(K_\lambda-\frac{K_x}{Ts+1}\right)-\frac{K_x}{Ts+1}\right]+\right.\\
&\quad\left.+K_\psi\right\}\xi(t)\,A_0\sin(\omega_0 t+\psi^*);\\
\lambda^* &= \frac{1}{Ts+1}\left(K_\lambda-\frac{K_x}{Ts+1}\right)\xi(t)\,A_0\sin(\omega_0 t+\psi^*);\\
x^* &= -\frac{K_x}{Ts+1}\xi(t)\,A_0\sin(\omega_0 t+\psi^*),
\end{aligned}\right\}\quad (X.100)$$

where

$$K=\frac{\alpha M_\Phi}{S_y};\quad K_\lambda=\frac{2\overline{K}^*_{\psi\lambda}}{\alpha N_0};\quad K_x=\frac{2\overline{K}^*_{\psi x}}{\alpha N_0};\quad K_\psi=\frac{2\overline{K}^*_{\psi\psi}}{N_0};\quad T=\frac{1}{\alpha}.$$

Equations (X.100) are simulated by the optimum receiver, the version of structural scheme of which is given in Fig. X.19. Optimum receiver is system FAPCh with the control along two channels.

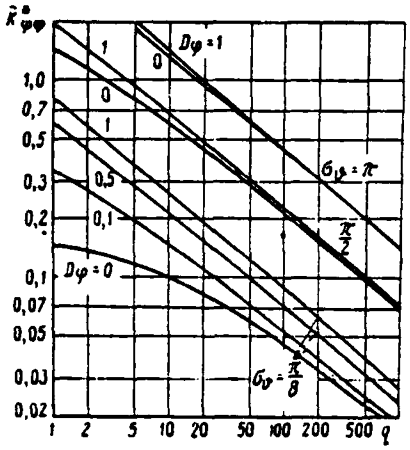



Fig. X.17. Dependence of the dispersion of phase error $\bar{K}^{*}_{\psi\psi}$ on the signal-to-noise ratio.

Page 319.

Frequency modulation. Let us record useful radio signal in the form

$$S(t, \lambda) = A_0 \cos[\omega_0 t + \psi(t)]; \qquad (X.101)$$
$$\psi(t) = \vartheta(t) + \varphi(t) = M_{ч} \int_0^t \lambda(\tau)\, d\tau + \varphi(t),$$

where $A_0$, $\omega_0$ - a priori known values of amplitude and frequency;

$\varphi(t)$ - component of the phase of signal, which is changed randomly due to the instability of the frequency of transmitter;

$M_{ч}$ $\sigma_{\omega}/\sigma_{\lambda}$ - constant coefficient (mutual conductance of the frequency shift key).

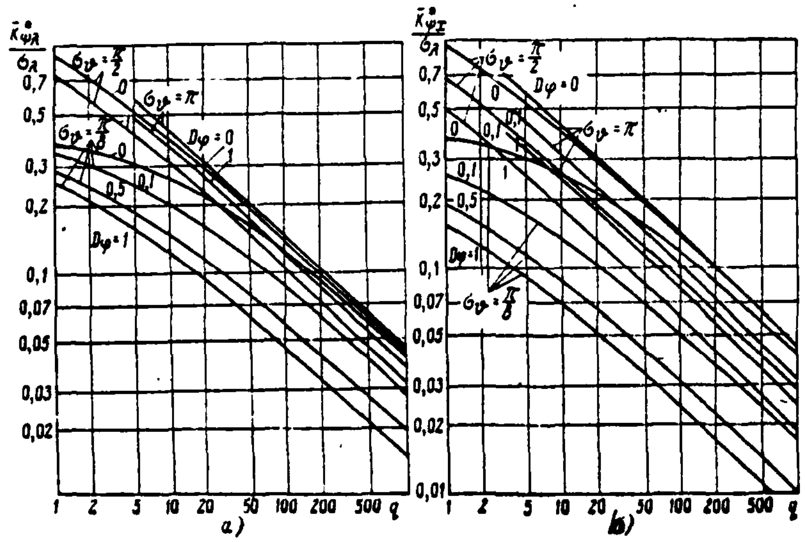


Fig. X.18. Dep·ndence of coefficient $\bar{K}^{*}_{\psi\lambda}/\sigma_{\lambda}$ on the signal-to-noise ratio.

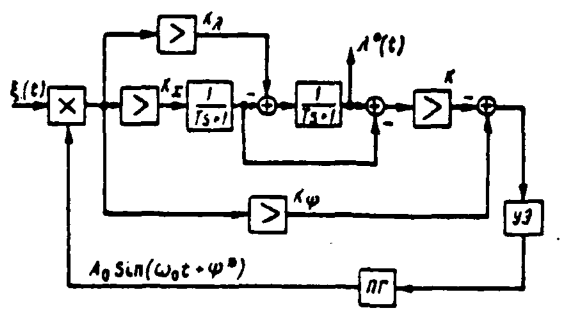


Fig. X.19. Structural scheme of the optimum receiver of phase-modulated radio signals.

Page 320.

The behavior of the random parameters of useful radio signal

(X.101) let us assign stochastic equations

$$\left.\begin{aligned} \dot{\psi} &= M_{\text{ч}}\lambda + \dot{\varphi}(t); \\ \dot{\lambda} &= -\alpha\lambda - \beta x + n_{\lambda}(t); \\ \dot{x} &= -\beta x + n_{\lambda}(t); \\ \dot{\varphi}(t) &= n_{\varphi}(t). \end{aligned}\right\} \quad (X.102)$$

In connection with radio signal (X.101) of the equation of optimum nonlinear filtration in the gaussian approximation/approach for the steady state after some simplifications they take the form

$$\left.\begin{aligned} 2M_{\text{ч}}\overline{K}^{*}_{\psi\lambda} + \frac{1}{2}N_{\varphi} + (\overline{K}^{*}_{\psi\psi})^{2}\overline{F}_{\psi\psi} &= 0; \\ 2\alpha\overline{K}^{*}_{\lambda\lambda} + 2\beta\overline{K}^{*}_{\lambda x} - \frac{1}{2}N_{\lambda} - (\overline{K}^{*}_{\psi\lambda})^{2}\overline{F}_{\psi\psi} &= 0; \\ 2\beta\overline{K}^{*}_{xx} - \frac{1}{2}N_{\lambda} - (\overline{K}^{*}_{\psi x})^{2}\overline{F}_{\psi\psi} &= 0; \\ M_{\text{ч}}\overline{K}^{*}_{\lambda\lambda} - \alpha\overline{K}^{*}_{\psi\lambda} - \beta\overline{K}^{*}_{\psi x} + \overline{K}^{*}_{\psi\lambda}\overline{K}^{*}_{\psi\psi}\overline{F}_{\psi\psi} &= 0; \\ M_{\text{ч}}\overline{K}^{*}_{\lambda x} - \beta\overline{K}^{*}_{\psi x} + \overline{K}^{*}_{\psi x}\overline{K}^{*}_{\psi\psi}\overline{F}_{\psi\psi} &= 0; \\ (\alpha+\beta)\overline{K}^{*}_{\lambda x} + \beta\overline{K}^{*}_{xx} - \frac{1}{2}N_{\lambda} - \overline{K}^{*}_{\psi\lambda}\overline{K}^{*}_{\psi x}\overline{F}_{\psi\psi} &= 0; \end{aligned}\right\} \quad (X.103)$$

$$\left.\begin{aligned} \dot{\psi}^{*} &= M_{\text{ч}}\lambda^{*} + \overline{K}^{*}_{\psi\psi}F_{\psi}; \\ \dot{\lambda}^{*} &= -\alpha\lambda^{*} - \beta x^{*} + \overline{K}^{*}_{\psi\lambda}F_{\psi}; \\ \dot{x} &= -\beta x^{*} + \overline{K}^{*}_{\psi\lambda}F_{\psi}, \end{aligned}\right\} \quad (X.104)$$

where

$$F_{\psi} = -\frac{2}{N_0}\xi(t)A_0\sin(\omega_0 t + \psi^{*}); \quad \overline{F}_{\psi\psi} = -\frac{A_0^2}{N_0}. \quad (X.105)$$

Let us convert system of equations (X.103). For this let us introduce into the examination the index of frequency modulation $\beta_{\text{чм}} = \sigma_{\omega}/\alpha$ and signal-to-noise ratio $q = A^2_0/2\alpha N_0$. Then with $\beta = \alpha$ we obtain

$$\left.\begin{aligned}
&2\beta_{\varphi M}\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda}+D_\varphi-2q\left(\overline{K}^{\bullet}_{\psi\psi}\right)^2=0;\\
&\overline{K}^{\bullet}_{\lambda\lambda}+\overline{K}^{\bullet}_{\lambda x}-2\sigma_\lambda^2+q\left(\overline{K}^{\bullet}_{\psi\lambda}\right)^2=0;\\
&\overline{K}^{\bullet}_{xx}-2\sigma_\lambda^2+q\left(\overline{K}^{\bullet}_{\psi x}\right)^2=0;
\end{aligned}\right\}\quad (X.106)$$

$$\left.\begin{aligned}
&\beta_{\varphi M}\frac{\overline{K}^{\bullet}_{\lambda\lambda}}{\sigma_\lambda}-\overline{K}^{\bullet}_{\psi\lambda}-\overline{K}^{\bullet}_{\psi x}-2q\overline{K}^{\bullet}_{\psi\psi}\overline{K}^{\bullet}_{\psi\lambda}=0;\\
&\beta_{\varphi M}\frac{\overline{K}^{\bullet}_{\lambda x}}{\sigma_\lambda}-\overline{K}^{\bullet}_{\psi x}-2q\overline{K}^{\bullet}_{\psi\psi}\overline{K}^{\bullet}_{\psi x}=0;\\
&2\overline{K}^{\bullet}_{\lambda x}+\overline{K}^{\bullet}_{xx}-4\sigma_\lambda^2+2q\overline{K}^{\bullet}_{\psi\lambda}\overline{K}^{\bullet}_{\psi x}=0.
\end{aligned}\right\}\quad (X.106)$$

Page 321.

Here $D_\varphi = N_\varphi/2\alpha$ - dispersion of random phase change of high-frequency oscillation for the time $\tau={}^1/\alpha$. For the transition/junction to the dimensionless quantities let us divide the fourth and fifth equations of system (X.106) on $\sigma_\lambda$, and the second, the third, and the sixth - on $\sigma_\lambda^2$, after which we will obtain

$$\left.\begin{aligned}
&2\beta_{\varphi M}\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda}+D_\varphi-2q\left(\overline{K}^{\bullet}_{\psi\psi}\right)^2=0;\\
&\delta^2_{\varphi M}+\frac{\overline{K}^{\bullet}_{\lambda x}}{\sigma_\lambda^2}-2+q\left(\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda}\right)^2=0;\\
&\frac{\overline{K}^{\bullet}_{xx}}{\sigma_\lambda^2}-2+q\left(\frac{\overline{K}^{\bullet}_{\psi x}}{\sigma_\lambda}\right)^2=0;\\
&\beta_{\varphi M}\delta^2_{\varphi M}-\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda}-\frac{\overline{K}^{\bullet}_{\psi x}}{\sigma_\lambda}-2q\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda}\overline{K}^{\bullet}_{\psi\psi}=0;\\
&\beta_{\varphi M}\frac{\overline{K}^{\bullet}_{\lambda x}}{\sigma_\lambda^2}-\frac{\overline{K}^{\bullet}_{\psi x}}{\sigma_\lambda}-2q\frac{\overline{K}^{\bullet}_{\psi x}}{\sigma_\lambda}\overline{K}^{\bullet}_{\psi\psi}=0;\\
&2\frac{\overline{K}^{\bullet}_{\lambda x}}{\sigma_\lambda^2}+\frac{\overline{K}^{\bullet}_{xx}}{\sigma_\lambda^2}-4+2q\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda}\cdot\frac{\overline{K}^{\bullet}_{\psi x}}{\sigma_\lambda}=0.
\end{aligned}\right\}\quad (X.107)$$

Here $\delta^2_{\varphi M}=\overline{K}^{\bullet}_{\lambda\lambda}/\sigma^2_\lambda$ the relative error of the filtration of informational communication/report $\lambda(t)$.

The results of solving the system of equations (X.107) numerical methods of TsVM for series/row of values $\beta_{\varphi M}$ and $D_\varphi$ depending on q (when $\overline{K}^{\bullet}_{\psi\psi}<2$ rad²) are given in Fig. X.20-X.22. Besides $\delta^2_{\varphi M}$ are indicated the values only of those coefficients, which are necessary for the construction of optimum receiver.

From the graph/diagram of the dependence of the relative error of filtration $\delta^2_{\varphi M}$ on the signal-to-noise ratio q, represented in Fig. X.20 it is apparent that with the fixed values of q and $\beta_{\varphi M}$ the error of filtration is minimum, when $D_\varphi=0$. With preset q and $D_\varphi$ the error

is reduced with an increase in the index of modulation $\beta_{ЧМ}$. The effect of the fluctuation of phase on $\delta^2_{ЧМ}$ is reduced in proportion to increase/growth $\beta_{ЧМ}$.

Page 322.

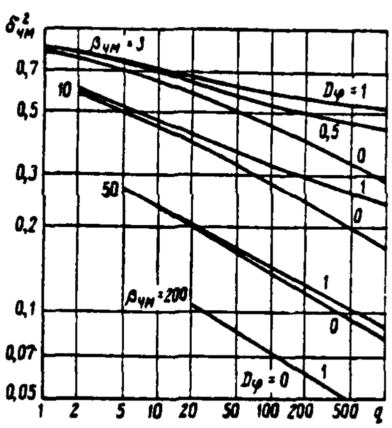



Fig X.20.

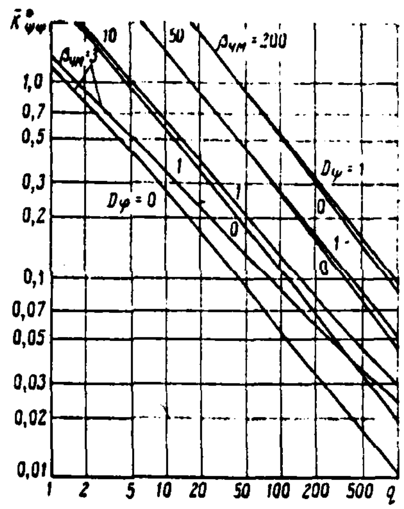



Fig. X.21.

Fig. X.20. Dependence of the square of the relative error of the filtration of communication/report on the signal-to-noise ratio with the frequency modulation.

Fig. X.21. Dependence of the dispersion of phase error $\bar{K}^{\bullet}_{\psi\psi}$ on the signal-to-noise ratio.

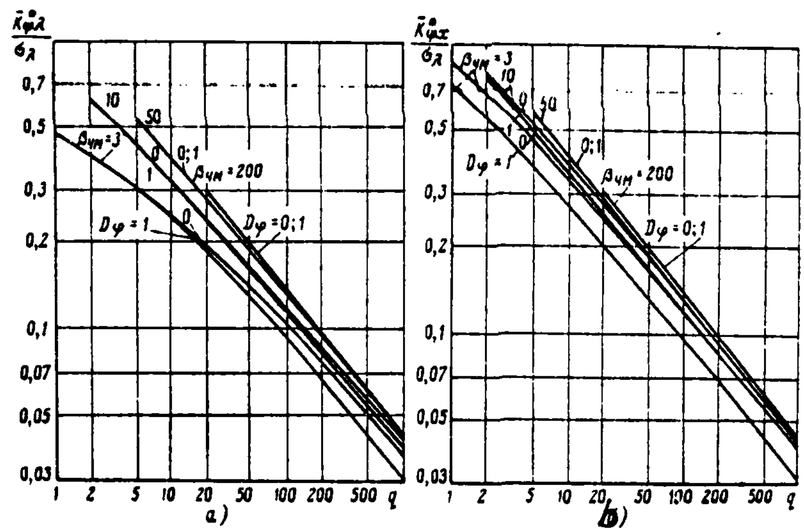



Fig. X.22.

Fig. X.22. Dependence of coefficients $\bar{K}^{*}_{\psi\lambda}/\sigma_{\lambda}$ and $\bar{K}^{*}_{\psi x}/\sigma_{\lambda}$ on the signal-to-noise ratio.

Page 323.

After substituting in equations (X.104) value $F_{\psi}$ from relationship/ratio (X.105), we will obtain the equations, which determine the structure of the optimum receiver of ChM radio signals:

$$\left.\begin{aligned} \dot{\psi} &= S_y\left[\frac{K}{Ts+1}\left(\frac{K_x}{Ts+1}-K_\lambda\right)-K_\psi\right]\xi(t)\,A_0\sin(\omega_0 t+\psi^*);\\ \lambda^* &= \frac{1}{Ts+1}\left[\frac{K_x}{Ts+1}-K_\lambda\right]\xi(t)\,A_0\sin(\omega_0 t+\psi^*);\\ x^* &= -\frac{K_x}{Ts+1}\,\xi(t)\,A_0\sin(\omega_0 t+\psi^*). \end{aligned}\right\} \quad (X.108)$$

Here $S_y$ - mutual conductance is control device;

$$K_\lambda = \frac{2\bar{K}^{*}_{\psi\lambda}}{\alpha N_0};\quad K_x = \frac{2\bar{K}^{*}_{\psi x}}{\alpha N_0};\quad K_\psi = \frac{2\bar{K}^{*}_{\psi\psi}}{N_0 S_y};\quad K = \frac{M_y}{S_y};\quad T = \frac{1}{\alpha}.$$

The structural scheme of the optimum receiver of ChM radio signals, simulating equation (X.108), is given in Fig. X.23. Optimum receiver is system FAPCh with the control along two channels. Actually it realizes the quasi-coherent perfecting of the oscillation accepted.

4. Complex modulation.

Amplitude-phase modulation by one-component Markov process. Let us consider the cases, when amplitude and phase of signal are modulated by one and the same random Markov process (communication/report) $\lambda(t)$.

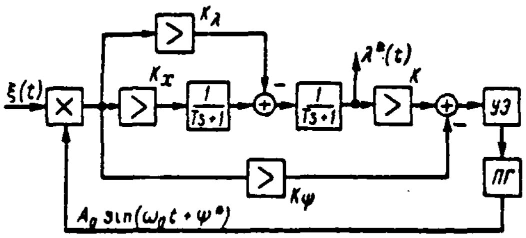



Fig. X.23. Structural scheme of the optimum receiver of the frequency modulated radio signals.

Page 324.

Useful signal with complex amplitude-phase (AM-FM) modulation can be represented in the form

$$S(t, \lambda) = M_A\lambda(t)\cos(\omega_0 t + \psi(t)); \qquad (X.109)$$
$$\psi(t) = \vartheta(t) + \varphi(t) = M_\Phi\lambda(t) + \varphi(t),$$

where $M_A$ and $M_\Phi$ - constant coefficients;

$\varphi(t)$ - the random walks of the phase of signal due to the instability of the frequency of the master oscillator.

Let for signal (X.109) the system of the a priori stochastic equations

$$\left.\begin{aligned} \dot{\psi} &= -M_\Phi\alpha\lambda + n_\vartheta(t) + \dot{\varphi}(t); \\ \dot{\lambda} &= -\alpha\lambda + n_\lambda(t); \\ \dot{\varphi}(t) &= n_\varphi(t), \end{aligned}\right\} \qquad (X.110)$$

where $n_\vartheta(t) = M_\Phi n_\lambda(t)$ be valid.

In this case the following expressions for the correlation and mutual-correlation functions will be valid:

$$\left.\begin{aligned} M[n_\lambda(t_1)n_\lambda(t_2)] &= \frac{1}{2}N_\lambda\delta(\tau);\\ M[n_\psi(t_1)n_\psi(t_2)] &= \frac{1}{2}M_\Phi^2N_\lambda\delta(\tau)+\frac{1}{2}N_\varphi\delta(\tau);\\ M[n_\varphi(t_1)n_\varphi(t_2)] &= \frac{1}{2}N_\varphi\delta(\tau);\\ M[n_\lambda(t_1)n_\psi(t_2)] &= \frac{1}{2}M_\Phi N_\lambda\delta(\tau), \end{aligned}\right\} \quad (X.111)$$

where

$$n_\psi(t) = M_\Phi n_\lambda(t) + n_\varphi(t).$$

For the case of the equation of optimum nonlinear filtration in question they will take the form

$$\left.\begin{aligned} \frac{1}{2}M_\Phi^2N_\lambda+\frac{1}{2}N_\varphi-2\alpha M_\Phi\overline{K}^*_{\psi\lambda}+(\overline{K}^*_{\psi\lambda})^2\overline{F}_{\lambda\lambda}+(\overline{K}^*_{\psi\psi})^2\overline{F}_{\psi\psi} &= 0;\\ \frac{1}{2}N_\lambda-2\alpha\overline{K}^*_{\lambda\lambda}+(\overline{K}^*_{\lambda\lambda})^2\overline{F}_{\lambda\lambda}+(\overline{K}^*_{\psi\lambda})^2\overline{F}_{\psi\psi} &= 0;\\ \frac{1}{2}M_\Phi N_\lambda-\alpha\overline{K}^*_{\psi\lambda}-\alpha M_\Phi\overline{K}^*_{\lambda\lambda}+\overline{K}^*_{\psi\lambda}\overline{K}^*_{\lambda\lambda}\overline{F}_{\lambda\lambda}+\overline{K}^*_{\psi\lambda}\overline{K}^*_{\psi\psi}\overline{F}_{\psi\psi} &= 0; \end{aligned}\right\} \quad (X.112)$$

$$\left.\begin{aligned} \dot{\psi}^* &= -\alpha M_\Phi\lambda^*+\overline{K}^*_{\psi\psi}F_\psi+\overline{K}^*_{\psi\lambda}F_\lambda;\\ \dot{\lambda}^* &= -\alpha\lambda^*+\overline{K}^*_{\lambda\lambda}F_\lambda+\overline{K}^*_{\psi\lambda}F_\psi. \end{aligned}\right\} \quad (X.113)$$

For signal (X.109) we find

$$\left.\begin{aligned} F_\psi &= -\frac{2}{N_0}\xi(t)M_A\lambda^*\sin(\omega_0t+\psi^*);\\ F_\lambda &= \frac{1}{N_0}[2\xi(t)M_A\cos(\omega_0t+\psi^*)-M_A^2\lambda^*];\\ \overline{F}_{\psi\psi} &= -\frac{M_A^2\sigma_\lambda^2}{N_0};\quad \overline{F}_{\lambda\lambda} = -\frac{M_A^2}{N_0}. \end{aligned}\right\} \quad (X.114)$$

Page 325.

Let us introduce signal-to-noise ratio $q = M_A^2\sigma_\lambda^2/2\alpha N_0$ and let us pass in the system of equations (X.112) to the dimensionless quantities. For this let us divide the second equation on $\sigma_\lambda^2$, and the third - on $\sigma_\lambda$. Then taking into account expressions (X.110-X.114) we obtain

$$\left.\begin{aligned} &2\sigma_\Phi^2 + D_\varphi - 2\sigma_\Phi \frac{\overline{K}^\bullet_{\psi\lambda}}{\sigma_\lambda} - 2q\left(\frac{\overline{K}^\bullet_{\psi\lambda}}{\sigma_\lambda}\right)^2 - 2q\,(\overline{K}^\bullet_{\psi\psi})^2 = 0;\\ &\delta^2_{AM-\Phi M} - 1 + q\delta^4_{AM-\Phi M} + q\,(K^\bullet_{\psi\psi})^2 \cdot 0;\\ &\frac{\overline{K}^\bullet_{\psi\lambda}}{\sigma_\lambda} - 2\sigma_\Phi + \sigma_\Phi\delta^2_{AM-\Phi M} + 2q\delta^2_{AM-\Phi M}\frac{\overline{K}^\bullet_{\psi\lambda}}{\sigma_\lambda} +\\ &+ 2q\frac{\overline{K}^\bullet_{\psi\lambda}}{\sigma_\lambda}\overline{K}^\bullet_{\psi\psi} = 0, \end{aligned}\right\} \quad (X.115)$$

where $\delta^2_{AM-\Phi M} = \frac{K^\bullet_{\lambda\lambda}}{\sigma_\lambda^2}$ - relative error of the filtration of communication/report; $D_\varphi = N_\varphi/2\alpha$ - dispersion of phase change for the time $^1/\alpha$.

The results of solving the system of equations (X.115) for

series/row of values $\sigma_\theta$ and $D_\varphi$ depending on q are given in Fig. X.24-X.26.

Fig. X.24 depicts the graph/diagram of the dependence of the square of the relative error of filtration $\delta^2_{\lambda M-\varphi M}$ on the signal-to-noise ratio. The dependences of coefficients $\overline{K}^{*}_{\psi\psi}$ and $\overline{K}^{*}_{\psi\lambda}/\sigma_\lambda$ on the signal-to-noise ratio are depicted in Fig. X.25, X.26.

After substituting into system (X.113) of value $F_\psi$ and $F_\lambda$ and expressions (X.114), we will obtain the equations, which determine the structure of the optimum sensing transducer AM-FM,:

$$\left.\begin{aligned} \dot{\psi}^* &= -S_y\{K_{\psi 1}\lambda^* + 2K_{\psi 2}\xi(t)M_A\lambda^*\sin(\omega_0 t+\psi^*) - \\ &\quad - K_{\psi\lambda 1}[2\xi(t)M_A\cos(\omega_0 t+\psi^*) - M_A^2\lambda^*]\}; \\ \lambda^* &= \frac{1}{Ts+1}\{K_\lambda[2\xi(t)M_A\cos(\omega_0 t+\psi^*) - M_A^2\lambda^*] - \\ &\quad - 2K_{\psi\lambda 2}\xi(t)M_A\lambda^*\sin(\omega_0 t+\psi^*)\}, \end{aligned}\right\} \quad (X.116)$$

where $S_y$ - mutual conductance of control device;

$$K_{\psi 1} = \frac{\alpha M_\Phi}{S_y}; \quad K_{\psi 2} = \frac{\overline{K}^*_{\psi\psi}}{S_y N_0}; \quad K_{\psi\lambda 1} = \frac{\overline{K}^*_{\psi\lambda}}{S_y N_0};$$

$$K_\lambda \quad \frac{\overline{K}^*_{\lambda\lambda}}{\alpha N_0}; \quad K_{\psi\lambda 2} = \frac{\overline{K}^*_{\psi\lambda}}{\alpha N_0}; \quad T - \frac{1}{\alpha}.$$

The structural scheme of the optimum sensing transducer AM-FM is depicted in Fig. X.27.

Page 326.

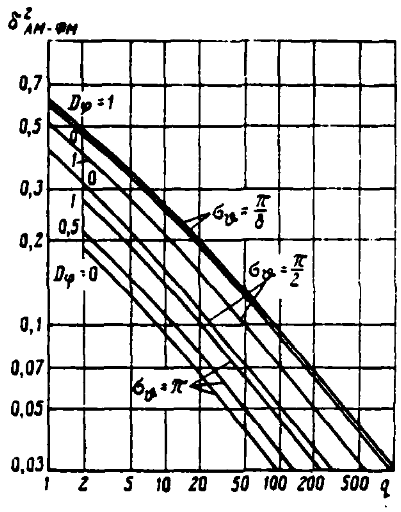


Fig. X.24.

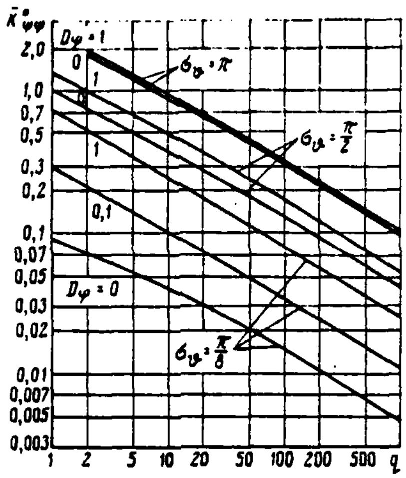


Fig. X.25.

Fig. X.24. Dependence of the square of the relative error of the filtration of communication/report on the signal-to-noise ratio with amplitude-phase modulation.

Fig. X.25. Dependence of the dispersion of phase error $\bar{K}^{*}_{\psi\psi}$ on the signal-to-noise ratio.

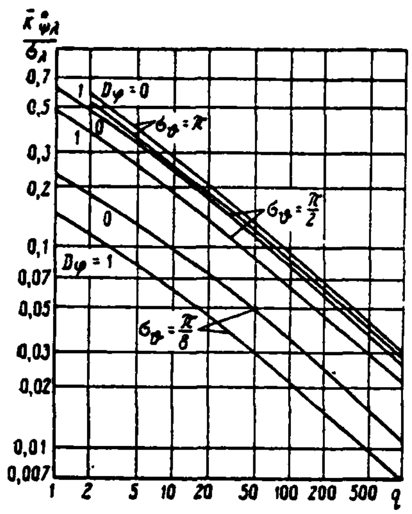


Fig. X.26.

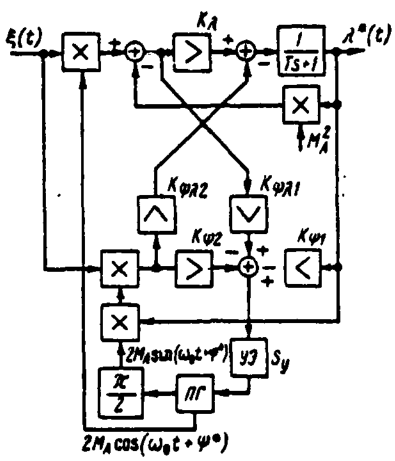


Fig. X.27.

Fig. X.26. Dependence of coefficient $\bar{K}^{*}_{\psi\lambda}/\sigma_{\lambda}$ on the signal-to-noise ratio.

Fig. X.27. Structural scheme of optimum receiver for the radio signal with amplitude-phase modulation.

Page 327.

Amplitude-frequency of modulation by one-component Markovian process. With complex amplitude-frequency (AM-ChM) modulation let us record useful radio signal in the form

$$S(t;\ \lambda) = M_A\lambda(t)\cos(\omega_0 t + \psi(t)); \qquad (X.117)$$

$$\psi(t) = \vartheta(t) + \varphi(t) = M_\psi\int_0^t \lambda(\tau)\,d\tau + \varphi(t),$$

where $M_A$ and $M_\psi$ - constant coefficients;

$\varphi(t)$ - the random walks of the phase of signal.

Let for signal (X.117) be valid the following system of a priori stochastic equations:

$$\left.\begin{aligned} \dot{\psi} &= M_\psi\lambda + \dot{\varphi}(t); \\ \dot{\lambda} &= -\alpha\lambda + n_\lambda(t); \\ \dot{\varphi} &= n_\varphi(t). \end{aligned}\right\} \qquad (X.118)$$

Let us record expressions for the correlation and mutual-correlation functions:

$$\left.\begin{aligned} M[n_\lambda(t_1)\,n_\lambda(t_2)] &= \frac{1}{2}N_\lambda\delta(\tau); \\ M[n_\varphi(t_1)\,n_\varphi(t_2)] &= \frac{1}{2}N_\varphi\delta(\tau). \end{aligned}\right\} \qquad (X.119)$$

In connection with AM-ChM radio signal (X.117) the equations of optimum nonlinear filtration take the form

$$\left.\begin{aligned} &\frac{1}{2} N_\varphi + 2M_{\text{Ч}}\overline{K}^{\bullet}_{\psi\lambda} + (\overline{K}^{\bullet}_{\psi\lambda})^2\overline{F}_{\lambda\lambda} + (\overline{K}^{\bullet}_{\psi\psi})^2\,\overline{F}_{\psi\psi} = 0; \\ &\frac{1}{2} N_\lambda - 2\alpha\overline{K}^{\bullet}_{\lambda\lambda} + (\overline{K}^{\bullet}_{\lambda\lambda})^2\,\overline{F}_{\lambda\lambda} + (\overline{K}^{\bullet}_{\psi\lambda})^2\,\overline{F}_{\psi\psi} = 0; \\ &M_{\text{Ч}}\overline{K}^{\bullet}_{\lambda\lambda} - \alpha\overline{K}^{\bullet}_{\psi\lambda} + \overline{K}^{\bullet}_{\psi\lambda}\overline{K}^{\bullet}_{\lambda\lambda}\overline{F}_{\lambda\lambda} + \overline{K}^{\bullet}_{\psi\psi}\overline{K}^{\bullet}_{\psi\lambda}\overline{F}_{\psi\psi} = 0; \end{aligned}\right\} \quad (X.120)$$

$$\left.\begin{aligned} \dot{\psi}^{\bullet} &= M_{\text{Ч}}\lambda^{\bullet} + \overline{K}^{\bullet}_{\psi\psi}F_\psi + \overline{K}^{\bullet}_{\psi\lambda}F_\lambda; \\ \dot{\lambda}^{\bullet} &= -\alpha\lambda^{\bullet} + \overline{K}^{\bullet}_{\lambda\lambda}F_\lambda + \overline{K}^{\bullet}_{\psi\lambda}F_\psi. \end{aligned}\right\} \quad (X.121)$$

Through the ordinary rules we find

$$\left.\begin{aligned} F_\psi &= -\frac{2}{N_0}\,\xi(t)\,M_A\lambda^{*}\sin(\omega_0 t + \psi^{*}); \\ F_\lambda &= \frac{1}{N_0}\,[2\xi(t)\,M_A\cos(\omega_0 t + \psi^{*}) - M_A^2\lambda^{*}]; \\ \overline{F}_{\psi\psi} &= -\frac{M_A^2\sigma_\lambda^2}{N_0}; \quad \overline{F}_{\lambda\lambda} = -\frac{M_A^2}{N_0}. \end{aligned}\right\} \quad (X.122)$$

Page 328.

For the transition/junction in the system of equations (X.120) to the dimensionless quantities let us introduce signal-to-noise ratio $q = M_A^2\sigma_\lambda^2/2\alpha N_0$, the index of frequency modulation $\beta_{\text{ЧМ}} = \sigma_\omega/\alpha$ and let us divide the second equation on $\sigma_\lambda^2$, and the third - on $\sigma_\lambda$. Then taking into account expressions (X.119)-(X.122) we obtain

$$\left.\begin{aligned} &2\beta_{\text{ЧМ}}\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda} + D_\varphi - 2q\,(\overline{K}^{\bullet}_{\psi\psi})^2 - 2q\left(\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda}\right)^2 = 0; \\ &1 - \delta^2_{AM-\text{ЧМ}} - q\left(\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda}\right)^2 - q\delta^4_{AM-\text{ЧМ}} = 0; \\ &\beta\delta^2_{AM-\text{ЧМ}} - \frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda} - 2q\overline{K}^{\bullet}_{\psi\psi}\,\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda} - 2q\,\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda}\,\delta^2_{AM-\text{ЧМ}} \quad 0; \end{aligned}\right| \quad (X.123)$$

where $\delta^2_{AM-ЧM} = \frac{\overline{K}^*_{\lambda\lambda}}{\sigma^2_\lambda}$ - relative error of the filtration of communication/report λ(t);

$D_\varphi = N_\varphi/2\alpha$ - dispersion of phase change for the time 1/α.

The results of solving the system of equations (X.123) numerical methods on TsVM for series/row of values $\beta_{ЧM}$ and $D_\varphi$ depending on q are given in Fig. X.28-X.30. Fig. x.28 depicts the graph/diagram of the dependence of the square of the relative error of filtration $\delta^2_{AM-ЧM}$, while Fig. X.29 and X.30 give the dependences of coefficients $\overline{K}^*_{\psi\lambda}/\sigma_\lambda$ and $\overline{K}^*_{\psi\psi}$, necessary for the construction of the optimum sensing transducer AM-ChM.

After substituting in system (X.121) values $F_\psi$ and $F_\lambda$ from expression (X.122), after transformations we will obtain the equations, which determine the structure of the optimum sensing transducer AM-ChM:

$$\left.\begin{aligned}\dot{\psi}^* &= S_y\{K_{\psi 1}\lambda^* - 2K_{\psi 2}\xi(t)M_A\lambda^*\sin(\omega_0 t + \psi^*) + \\ &\quad + K_{\psi\lambda 1}[2\xi(t)M_A\cos(\omega_0 t + \psi^*) - M_A^2\lambda^*]\};\\ \lambda^* &= \frac{1}{Ts+1}\{K_\lambda[2\xi(t)M_A\cos(\omega_0 t + \psi^*) - M_A^2\lambda^*] - \\ &\quad - 2K_{\psi\lambda 2}\xi(t)M_A\lambda^*\sin(\omega_0 t + \psi^*)\};\end{aligned}\right\} \quad (X.124)$$

here $S_y$ - mutual conductance of control device;

$$K_{\psi 1} = \frac{M_\psi}{S_y}; \quad K_{\psi 2} = \frac{\overline{K}^{\bullet}_{\psi\psi}}{S_y N_0}; \quad K_{\psi\lambda 1} = \frac{\overline{K}^{\bullet}_{\psi\lambda}}{S_y N_0}; \; ..$$

$$K_\lambda = \frac{\overline{K}^{\bullet}_{\lambda\lambda}}{\alpha N_0}; \quad K_{\psi\lambda 2} = \frac{\overline{K}^{\bullet}_{\psi\lambda}}{\alpha N_0}; \quad T = \frac{1}{\alpha}.$$

The structural scheme of the optimum sensing transducer AM-ChM is depicted in Fig. X.31.

Page 329.

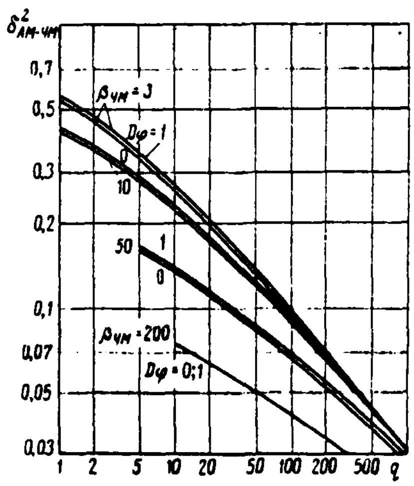



Fig. X.28

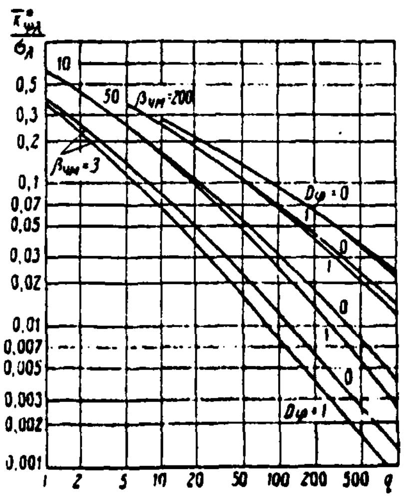
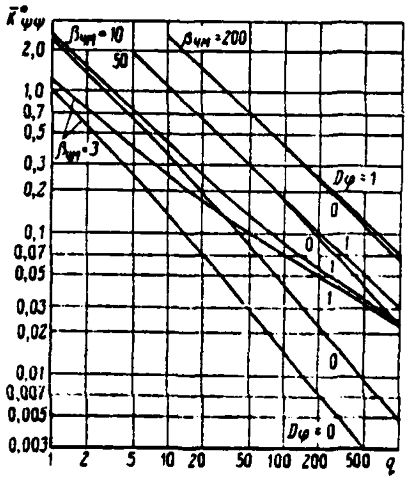



Fig. X.29.

Fig. X.28. Dependence of the square of the relative error of filtration on the signal-to-noise ratio with amplitude-frequency modulation.

Fig. X.29. Dependence of coefficient $\overline{K}^{*}_{\psi\lambda}/\sigma_{\lambda}$ on the signal-to-noise ratio.



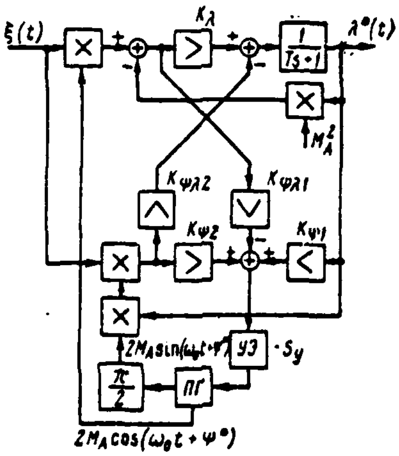


Fig. X.30. Fig. X.31.

Fig. X.30. Dependence of the dispersion of phase error $K^{*}_{\psi\psi}$ on the signal-to-noise ratio.

Fig. X.31. Structural scheme of optimum receiver for the radio signal with amplitude-frequency modulation.

Page 330.

Amplitude-phase modulation by two-component Markov process. With amplitude-phase (AM-FM) modulation useful radio signal takes the form

$$S(t, \lambda) = M_A\lambda(t)\cos(\omega_0 t + \psi(t)); \qquad (X.125)$$
$$\psi(t) = \vartheta(t) + \varphi(t) = M_\Phi\lambda(t) + \varphi(t),$$

where $M_\Phi = \sigma_\theta/\sigma_\lambda$ - constant coefficient (mutual conductance of phase modulator);

$M_A$ - constant coefficient;

φ(t) - the random walks of the phase of signal.

The behavior of the random parameters of radio signal (X.125) is described stochastic equations

$$\left.\begin{aligned} \dot{\psi} &= M_\Phi \dot{\lambda} + \dot{\varphi}(t); \\ \dot{\lambda} &= -\alpha\lambda - \beta x + n_\lambda(t); \\ \dot{x} &= -\beta x + n_\lambda(t); \\ \dot{\varphi} &= n_\varphi(t). \end{aligned}\right\} \qquad (X.126)$$

In this case will be valid the following expressions for the correlation and mutual-correlation functions:

$$\left.\begin{aligned} M[n_\lambda(t_1)\,n_\lambda(t_2)] &= \frac{1}{2}N_\lambda\delta(\tau); \\ M[n_\psi(t_1)\,n_\psi(t_2)] &= \frac{1}{2}M_\Phi^2 N_\lambda\delta(\tau) + \frac{1}{2}N_\varphi\delta(\tau); \\ M[n_\varphi(t_1)\,n_\varphi(t_2)] &= \frac{1}{2}N_\varphi\delta(\tau); \\ M[n_\lambda(t_1)\,n_\psi(t_2)] &= \frac{1}{2}M_\Phi N_\lambda\delta(\tau); \\ n_\psi(t) &= M_\Phi n_\lambda(t) + n_\varphi(t). \end{aligned}\right\} \qquad (X.127)$$

In connection with this case the equations of optimum nonlinear filtration take the form

$$\left.\begin{aligned}&2\alpha M_\psi \overline{K}^{\bullet}_{\psi\lambda}+2\beta M_\psi \overline{K}^{\bullet}_{\psi x}-\frac{1}{2}M_\psi^2 N_\lambda-\frac{1}{2}N_\psi+(\overline{K}^{\bullet}_{\psi\psi})^2\,\overline{F}_{\psi\psi}+\\&\quad+(\overline{K}^{\bullet}_{\psi\lambda})^2\,\overline{F}_{\lambda\lambda}=0;\\&2\alpha\overline{K}^{\bullet}_{\lambda\lambda}+2\beta\overline{K}^{\bullet}_{\lambda x}-\frac{1}{2}N_\lambda-(\overline{K}^{\bullet}_{\psi\lambda})^2\,\overline{F}_{\psi\psi}-(\overline{K}^{\bullet}_{\lambda\lambda})^2\overline{F}_{\lambda\lambda}=0;\\&2\beta\overline{K}^{\bullet}_{xx}-\frac{1}{2}N_\lambda-(\overline{K}^{\bullet}_{\psi x})^2\,\overline{F}_{\psi\psi}-(\overline{K}^{\bullet}_{\lambda x})\,\overline{F}_{\lambda\lambda}=0;\end{aligned}\right\}\quad(X.128)$$

$$\left.\begin{aligned}&\alpha M_\psi\overline{K}^{\bullet}_{\lambda\lambda}+\beta M_\psi\overline{K}^{\bullet}_{\lambda x}+\alpha\overline{K}^{\bullet}_{\psi\lambda}+\beta\overline{K}^{\bullet}_{\psi x}-\frac{1}{2}M_\psi N_\lambda-\\&\quad-\overline{K}^{\bullet}_{\psi\psi}\overline{K}^{\bullet}_{\psi\lambda}\overline{F}_{\psi\psi}-\overline{K}^{\bullet}_{\lambda\lambda}\overline{K}^{\bullet}_{\psi\lambda}\overline{F}_{\lambda\lambda}=0;\\&\alpha M_\psi\overline{K}^{\bullet}_{\lambda x}+\beta M_\psi\overline{K}^{\bullet}_{xx}+\beta\overline{K}^{\bullet}_{\psi x}-\frac{1}{2}M_\psi N_\lambda-\overline{K}^{\bullet}_{\psi\psi}\overline{K}^{\bullet}_{\psi x}\overline{F}_{\psi\psi}-\\&\quad-\overline{K}^{\bullet}_{\psi\lambda}\overline{K}^{\bullet}_{\lambda x}\overline{F}_{\lambda\lambda}=0;\\&\alpha\overline{K}^{\bullet}_{\lambda x}+\beta\overline{K}^{\bullet}_{xx}+\beta\overline{K}^{\bullet}_{\lambda x}-\frac{1}{2}N_\lambda-\overline{K}^{\bullet}_{\psi\lambda}K^{\bullet}_{\psi x}\overline{F}_{\psi\psi}-\overline{K}^{\bullet}_{\lambda\lambda}\overline{K}^{\bullet}_{\lambda x}\overline{F}_{\lambda\lambda}=0;\end{aligned}\right\}\quad(X.128)$$

$$\left.\begin{aligned}\dot{\psi}^{\bullet}&=M_\psi\lambda^{\bullet}+\overline{K}^{\bullet}_{\psi\psi}F_\psi+\overline{K}^{\bullet}_{\psi\lambda}F_\lambda;\\\dot{\lambda}^{\bullet}&=-\alpha\lambda^{\bullet}-\beta x^{\bullet}+\overline{K}^{\bullet}_{\psi\lambda}F_\psi+\overline{K}^{\bullet}_{\lambda\lambda}F_\lambda;\\\dot{x}^{\bullet}&=-\beta x^{\bullet}+\overline{K}^{\bullet}_{\psi x}F_\psi+\overline{K}^{\bullet}_{\lambda x}F_\lambda.\end{aligned}\right\}\quad(X.129)$$

Page 331.

For signal (X.125) we have

$$\left.\begin{aligned}F_\psi&=-\frac{2}{N_0}\xi(t)\,M_A\lambda^{*}\sin(\omega_0 t+\psi^{*});\\F_\lambda&=\frac{1}{N_0}[2\xi(t)\,M_A\cos(\omega_0 t+\psi^{*})-M_A^2\lambda^{*}];\\\overline{F}_{\psi\psi}&=-\frac{M_A^2\sigma_\lambda^2}{N_0};\\\overline{F}_{\lambda\lambda}&=-\frac{M_A^2}{N_0}.\end{aligned}\right\}\quad(X.130)$$

For the transition/junction in the system of equations (X.128) to the dimensionless quantities let us introduce into the examination

signal-to-noise ratio q= $M_\lambda^2\sigma_\lambda^2/2\alpha N_0$ and let us divide the fourth and fifth equations into $\sigma_\lambda$, and the second, the third and the sixth - on $\sigma_\lambda^2$. After this taking into account relationships/ratios (X.127-X.130) with $\beta=\alpha$ we will obtain

$$\left.\begin{aligned}
&2\sigma_\vartheta\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda}+2\sigma_\vartheta\frac{\overline{K}^{\bullet}_{\psi x}}{\sigma_\lambda}-4\sigma_\vartheta^2-D_\varphi+2q\left(\overline{K}^{\bullet}_{\psi\psi}\right)^2+2q\left(\overline{K}^{\bullet}_{\psi\lambda}\right)^2=0;\\
&\delta^2_{AM-\Phi M}+\frac{\overline{K}^{\bullet}_{\lambda x}}{\sigma_\lambda^2}-2+q\left(\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda}\right)^2+q\delta^4_{AM-\Phi M}=0;\\
&\frac{\overline{K}^{\bullet}_{xx}}{\sigma_\lambda^2}-2+q\left(\frac{\overline{K}^{\bullet}_{\psi x}}{\sigma_\lambda}\right)^2+q\left(\frac{\overline{K}^{\bullet}_{\lambda x}}{\sigma_\lambda^2}\right)^2=0;\\
&\sigma_\vartheta\delta^2_{AM-\Phi M}+\sigma_\vartheta\frac{\overline{K}^{\bullet}_{\lambda x}}{\sigma_\lambda^2}+\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda}+\frac{\overline{K}^{\bullet}_{\psi x}}{\sigma_\lambda}-4\sigma_\vartheta+\\
&\quad+2q\overline{K}^{\bullet}_{\psi\psi}\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda}+2q\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda}\delta^2_{AM-\Phi M}=0;
\end{aligned}\right|\quad (X.131)$$

$$\left.\begin{aligned}
&\sigma_\vartheta\frac{\overline{K}^{\bullet}_{\lambda x}}{\sigma_\lambda^2}+\sigma_\vartheta\frac{\overline{K}^{\bullet}_{xx}}{\sigma_\lambda^2}+\frac{\overline{K}^{\bullet}_{\psi x}}{\sigma_\lambda}-4\sigma_\vartheta+\\
&\quad+2q\overline{K}^{\bullet}_{\psi\psi}\frac{\overline{K}^{\bullet}_{\psi x}}{\sigma_\lambda}+2q\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda}\cdot\frac{\overline{K}^{\bullet}_{\lambda x}}{\sigma_\lambda^2}=0;\\
&2\frac{\overline{K}^{\bullet}_{\lambda x}}{\sigma_\lambda^2}+\frac{\overline{K}^{\bullet}_{xx}}{\sigma_\lambda^2}-4+2q\frac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_\lambda}\frac{\overline{K}^{\bullet}_{\psi x}}{\sigma_\lambda}+2q\frac{\overline{K}^{\bullet}_{\lambda x}}{\sigma_\lambda^2}\delta^2_{AM-\Phi M}=0;
\end{aligned}\right|\quad (X.131)$$

here $\delta^2_{AM-\Phi M}=\overline{K}^{\bullet}_{\lambda\lambda}/\sigma_\lambda^2$ - the relative error of the filtration of communication/report $\lambda^*(t)$; $D_\varphi=N_\varphi/2\alpha$ - the dispersion of phase change for the time $^1/\alpha$.

Results of the solution of system of equations (X.131) numerical methods on TsVM for series/row of values $\sigma_\vartheta$ and $D_\varphi$ depending on q are given in Fig. X.32-X.34. Fig. X.32 depicts the dependence of the relative error of filtration $\delta^2_{AM-\Phi M}$, while Fig. X.33-X.34 gives the graph/diagrams of the dependence of coefficients $\overline{K}^{\bullet}_{\lambda x}/\sigma_\lambda^2$, $\overline{K}^{\bullet}_{\psi\lambda}/\sigma_\lambda$, $\overline{K}^{\bullet}_{\psi x}/\sigma_\lambda$, $\overline{K}^{\bullet}_{\psi\psi}$, necessary for the construction of the optimum sensing transducer AM-FM.

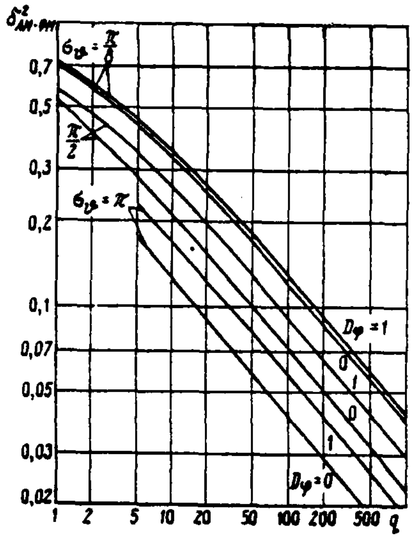



Fig. X.32.

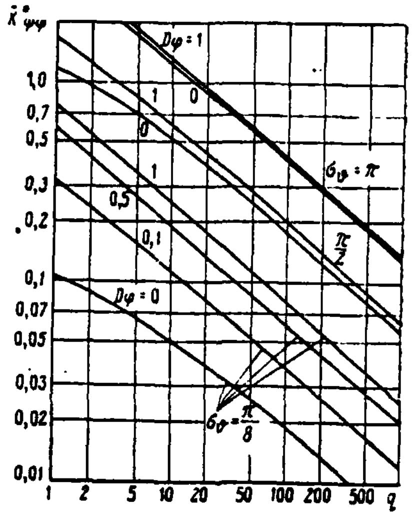



Fig. X.33.

Fig. X.32. Dependence of the square of the relative error of the filtration of communication/report on the signal-to-noise ratio with amplitude-phase modulation.

Fig. X.33. Dependence of the dispersion of phase error $\bar{K}^{*}_{\psi\psi}$ on the signal-to-noise ratio.

Page 333.

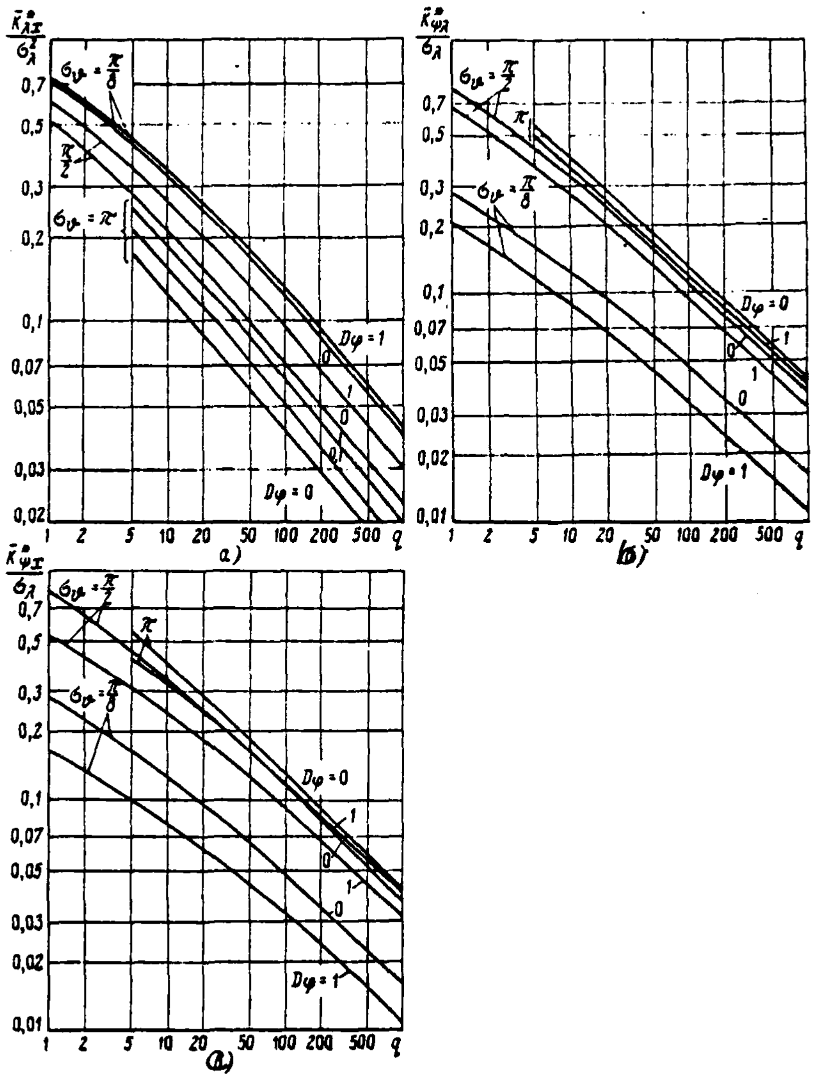



Fig. X.34. Dependence of coefficients $\overline{K}^{*}_{\lambda x}/\sigma^{2}_{\lambda}$, $\overline{K}^{*}_{\psi\lambda}/\sigma_{\lambda}$ and $\overline{K}^{*}_{\psi x}/\sigma_{\lambda}$ on the signal-to-noise ratio.

Page 334.

After substituting into system (X.129) of value $F_\psi$ and $F_\lambda$, determined according to formulas (X.130), after transformations we will obtain the equations, which determine the structure of the optimum receiver:

$$\left.\begin{aligned} &\dot{\psi} - S_y\{-K\lambda^* - Kx^* - K_\psi \xi(t) M_A \lambda^* \sin(\omega_0 t + \psi^*) + \\ &\quad + K_{\psi\lambda 1}[2\xi(t) M_A \cos(\omega_0 t + \psi^*) - M_A^2 \lambda^*]\}; \\ &\lambda^* - \frac{1}{Ts+1}\{-x^* - K_{\psi\lambda 2}\xi(t) M_A \lambda^* \sin(\omega_0 t + \psi^*) + \\ &\quad + K_\lambda[2\xi(t) M_A \cos(\omega_0 t + \psi^*) - M_A^2 \lambda^*]\}; \\ &x^* = \frac{1}{Ts+1}\{K_{\lambda x}[2\xi(t) M_A \cos(\omega_0 t + \psi^*) - M_A^2 \lambda^*] - \\ &\quad - K_{\psi x}\xi(t) M_A \lambda^* \sin(\omega_0 t + \psi^*)\}. \end{aligned}\right\} \quad (X.132)$$

Here

$$K = \frac{M_{\varphi}\alpha}{S_y}; \quad K_\psi = \frac{2\overline{K}^*_{\psi\psi}}{S_y N_0}; \quad K_{\psi\lambda 1} = \frac{\overline{K}^*_{\psi\lambda}}{S_y N_0}; \quad K_{\psi\lambda 2} = \frac{2\overline{K}^*_{\psi\lambda}}{\alpha N_0};$$

$$K_\lambda = \frac{\overline{K}^*_{\lambda\lambda}}{\alpha N_0}; \quad K_{\psi x} = \frac{2\overline{K}^*_{\psi x}}{\alpha N_0}; \quad K_{\lambda x} = \frac{\overline{K}^*_{\lambda x}}{\alpha N_0}; \quad T = \frac{1}{\alpha}.$$

The version of the structural scheme of the optimum sensing transducer AM-FM is given in Fig. X.35.

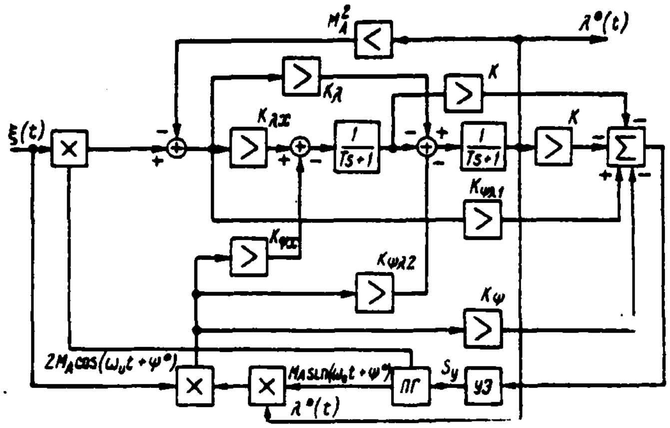



Fig. X.35. Structural scheme of optimum receiver for the radio signal with amplitude-phase modulation.

Page 335.

Amplitude-frequency modulation by two-component Markov process. With amplitude-frequency (AM-ChM) modulation useful radio signal let us record in the following form:

$$S(t, \lambda) = M_A \lambda(t) \cos(\omega_0 t + \psi(t)); \qquad \text{(X.133)}$$

$$\psi(t) = \vartheta(t) + \varphi(t) = M_\Psi \int_0^t \lambda(\tau)\, d\tau + \varphi(t),$$

where $M_A$ and $M_\Psi$ - constant coefficients; $\varphi$(t) - the random walks of the phase of signal.

We assume that for signal (X.133) the system of the a priori stochastic equations

$$
\left.\begin{aligned}
\dot{\psi} &= M_q\lambda + \dot{\varphi}(t);\\
\dot{\lambda} &= -\alpha\lambda - \beta x + n_\lambda(t);\\
\dot{x} &= -\beta x + n_\lambda(t);\\
\dot{\varphi}(t) &= n_\varphi(t).
\end{aligned}\right\} \qquad (X.134)
$$

is valid.

In this case will be valid the following expressions for the correlation functions:

$$
\left.\begin{aligned}
M[n_\lambda(t_1)\,n_\lambda(t_2)] &= \frac{1}{2}N_\lambda\delta(\tau);\\
M[n_\varphi(t_1)\,n_\varphi(t_2)] &= \frac{1}{2}N_\varphi\delta(\tau).
\end{aligned}\right\} \qquad (X.135)
$$

In connection with AM-ChM radio signal (X.133) the equations of optimum nonlinear filtration take the form

$$
\left.\begin{aligned}
&\frac{1}{2}N_\varphi + 2M_q\overline{K}^*_{\psi\lambda} + (\overline{K}^*_{\psi\psi})^2\,\overline{F}_{\psi\psi} + (\overline{K}^*_{\psi\lambda})^2\,\overline{F}_{\lambda\lambda} = 0;\\
&2\alpha\overline{K}^*_{\lambda\lambda} + 2\beta\overline{K}^*_{\lambda x} - \frac{1}{2}N_\lambda - (\overline{K}^*_{\psi\lambda})^2\,\overline{F}_{\psi\psi} - (\overline{K}^*_{\lambda\lambda})^2\,\overline{F}_{\lambda\lambda} = 0;\\
&2\beta\overline{K}^*_{xx} - \frac{1}{2}N_\lambda - (\overline{K}^*_{\psi x})^2\,\overline{F}_{\psi\psi} - (\overline{K}^*_{\lambda x})^2\,\overline{F}_{\lambda\lambda} = 0;\\
&M_q\overline{K}^*_{\lambda\lambda} - \alpha\overline{K}^*_{\psi\lambda} - \beta\overline{K}^*_{\psi x} + \overline{K}^*_{\psi\lambda}\overline{K}^*_{\psi\psi}\overline{F}_{\psi\psi} + \overline{K}^*_{\psi\lambda}\overline{K}^*_{\lambda\lambda}\overline{F}_{\lambda\lambda} = 0;\\
&M_q\overline{K}^*_{\lambda x} - \beta\overline{K}^*_{\psi x} + \overline{K}^*_{\psi x}\overline{K}^*_{\psi\psi}\overline{F}_{\psi\psi} + \overline{K}^*_{\psi\lambda}\overline{K}^*_{\lambda x}\overline{F}_{\lambda\lambda} = 0;\\
&(\alpha+\beta)\,\overline{K}^*_{\lambda x} + \beta\overline{K}^*_{xx} - \frac{1}{2}N_\lambda - \overline{K}^*_{\psi\lambda}\overline{K}^*_{\psi x}\overline{F}_{\psi\psi} - \overline{K}^*_{\lambda\lambda}\overline{K}^*_{\lambda x}\overline{F}_{\lambda\lambda} = 0;
\end{aligned}\right\} \qquad (X.136)
$$

$$
\left.\begin{aligned}
\dot{\psi}^* &= M_q\lambda^* + \overline{K}^*_{\psi\psi}F_\psi + \overline{K}^*_{\psi\lambda}F_\lambda;\\
\dot{\lambda}^* &= -\alpha\lambda^* - \beta x^* + \overline{K}^*_{\lambda\lambda}F_\lambda + \overline{K}^*_{\psi\lambda}F_\psi;\\
\dot{x}^* &= -\beta x^* + \overline{K}^*_{\lambda x}F_\lambda + \overline{K}^*_{\psi x}F_\psi.
\end{aligned}\right\} \qquad (X.137)
$$

Page 336.

For signal (X.133) let us find

$$\left.\begin{aligned} F_\psi &= -\frac{2}{N_0}\xi(t)\,M_A\lambda^*\sin(\omega_0 t+\psi^*);\\ F_\lambda &= \frac{1}{N_0}\left[2\xi(t)\,M_A\cos(\omega_0 t+\psi^*)-M_A^2\lambda^*\right];\\ \overline{F}_{\psi\psi} &= -\frac{M_A^2\sigma_\lambda^2}{N_0};\\ \overline{F}_{\lambda\lambda} &= -\frac{M_A^2}{N_0}. \end{aligned}\right\} \quad (X.138)$$

For the transition/junction in the system of equations (X.136) to the dimensionless quantities let us introduce into the examination the index of frequency modulation $\beta_{\text{чм}}=\sigma_\omega/\alpha$, signal-to-noise ratio $q=M_A^2\sigma_\lambda^2/2\alpha N_0$ and let us divide the fourth and fifth equations of system on $\sigma_\lambda$, and the second, the third and the sixth - on $\sigma_\lambda^2$. After this taking into account formulas (X.135)-(X.138) we will obtain with $\beta=\alpha$

$$\left.\begin{array}{c}
\beta_{\text{ЧМ}}\dfrac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_{\lambda}}+D_{\varphi}-2q\,(K_{\psi\psi})^{2}-2q\left(\dfrac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_{\lambda}}\right)^{2}=0;\\
2\delta^{2}_{AM-\text{ЧМ}}+\dfrac{\overline{K}^{\bullet}_{\lambda x}}{\sigma^{2}_{\lambda}}-2+q\left(\dfrac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_{\lambda}}\right)^{2}+q\delta^{4}_{AM-\text{ЧМ}}=0;\\
\dfrac{\overline{K}^{\bullet}_{xx}}{\sigma^{2}_{\lambda}}-2+q\left(\dfrac{\overline{K}^{\bullet}_{\psi x}}{\sigma_{\lambda}}\right)^{2}+q\left(\dfrac{\overline{K}^{\bullet}_{\lambda x}}{\sigma^{2}_{\lambda}}\right)^{2}=0;\\
\beta_{\text{ЧМ}}\delta^{2}_{AM-\text{ЧМ}}-2\dfrac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_{\lambda}}-2\dfrac{\overline{K}^{\bullet}_{\psi x}}{\sigma_{\lambda}}-4q\overline{K}^{\bullet}_{\psi\psi}\dfrac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_{\lambda}}-\\
-4q\dfrac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_{\lambda}}\delta^{2}_{AM-\text{ЧМ}}=0;\\
\beta_{\text{ЧМ}}\dfrac{\overline{K}^{\bullet}_{\lambda x}}{\sigma^{2}_{\lambda}}-2\dfrac{\overline{K}^{\bullet}_{\psi x}}{\sigma_{\lambda}}-4q\overline{K}^{\bullet}_{\psi\psi}\dfrac{\overline{K}^{\bullet}_{\psi x}}{\sigma_{\lambda}}-4q\dfrac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_{\lambda}}\dfrac{\overline{K}^{\bullet}_{\lambda x}}{\sigma^{2}_{\lambda}}=0;\\
2\dfrac{\overline{K}^{\bullet}_{\lambda x}}{\sigma^{2}_{\lambda}}+\dfrac{\overline{K}^{\bullet}_{xx}}{\sigma^{2}_{\lambda}}-4+2q\dfrac{\overline{K}^{\bullet}_{\psi\lambda}}{\sigma_{\lambda}}\dfrac{\overline{K}^{\bullet}_{\psi x}}{\sigma_{\lambda}}+2q\dfrac{\overline{K}^{\bullet}_{\lambda x}}{\sigma^{2}_{\lambda}}\delta^{2}_{AM-\text{ЧМ}}=0,
\end{array}\right\}\quad (X.139)$$

where $\delta^{2}_{AM-\text{ЧМ}}=\dfrac{\overline{K}^{\bullet}_{\lambda\lambda}}{\sigma^{2}_{\lambda}}$ - relative error of the filtration of communication/report $\lambda^{*}(t)$; $D_{\varphi}=N_{\varphi}/2\alpha$ - the dispersion of phase change for the time $1/\alpha$.

The results of solving the system of equations (X.139) on TsVM for the series/row of values $\beta_{\text{ЧМ}}$ and $D_{\varphi}$ depending on q are given numerical methods in Fig. X.36-X.38. Fig. X.36 depicts the graph/diagram of the dependence of the relative error of filtration $\delta^{2}_{AM-\text{ЧМ}}$, while Fig. X.37-X.38 gives the dependences of coefficients $\overline{K}^{\bullet}_{\lambda x}/\sigma^{2}_{\lambda}$, $\overline{K}^{\bullet}_{\psi\lambda}/\sigma_{\lambda}$, $K^{\bullet}_{\psi\psi}$, $\overline{K}^{\bullet}_{\psi x}/\sigma_{\lambda}$ necessary for the construction of the optimum sensing transducer AM-ChM.

Page 337.

Let us substitute in system (X.137) of value $F_{\psi}$ and $F_{\lambda}$, found from formulas (X.138), after transformations we will obtain the

equations, which determine the structure of the optimum receiver:

$$\left.\begin{aligned}\dot{\psi} &= S_y\{K\lambda^* - K_\psi \xi(t) M_A \lambda^* \sin(\omega_0 t + \psi^*) + K_{\psi\lambda 1}\times\\ &\times [2\xi(t) M_A \cos(\omega_0 t + \psi^*) - M_A^2 \lambda^*]\};\\ \lambda^* &= \frac{1}{Ts+1}\left\{\left(K_\lambda - \frac{K_{\lambda x}}{Ts+1}\right)[2\xi(t) M_A \cos(\omega_0 t + \psi^*) - \right.\\ &\left. - M_A^2\lambda^*] + \left(\frac{K_{\psi x}}{Ts+1} + K_{\psi\lambda 2}\right) \xi(t) M_A \lambda^* \sin(\omega_0 t + \psi^*)\right\}.\end{aligned}\right\} \quad (X.140)$$

Here

$$K = \frac{M_y}{S_y};\quad K_\lambda = \frac{K^*_{\lambda\lambda}}{\alpha N_0};\quad K_{\lambda x} = \frac{K^*_{\lambda x}}{\alpha N_0};\quad K_{\psi x} = \frac{2\bar{K}^*_{\psi x}}{\alpha N_0};\quad K_{\psi\lambda 1} = \frac{K^*_{\psi\lambda}}{S_y N_0};$$

$$K_{\psi\lambda 2} = \frac{2\bar{K}^*_{\psi\lambda}}{\alpha N_0};\quad K_\psi = \frac{2\bar{K}^*_{\psi\psi}}{S_y N_0};\quad T = \frac{1}{\alpha}.$$

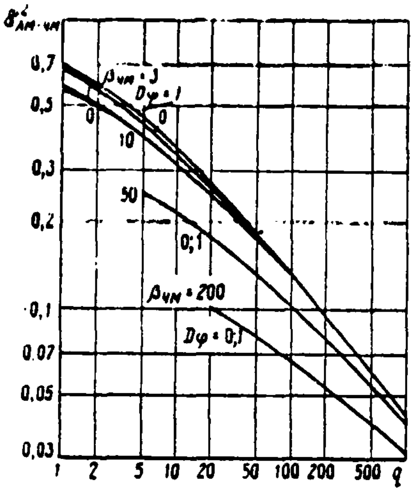



Fig. X.36.

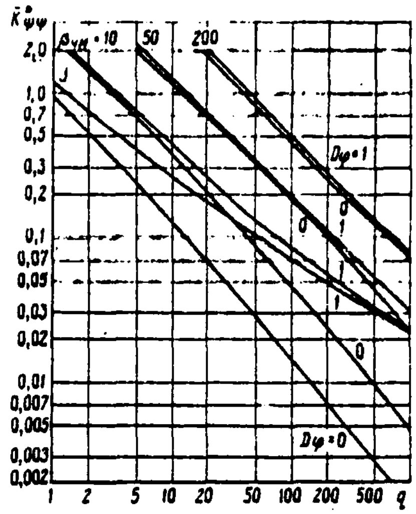



Fig. X.37.

Fig. X.36. Dependence of the square of the relative error of filtration of communication/report on the signal-to-noise ratio with amplitude-frequency modulation.

Fig. X.37. Dependence of the dispersion of phase error $\bar{K}^{*}_{\psi\psi}$ on the signal-to-noise ratio.

Page 338.

The structural scheme of the optimum sensing transducer AM-ChM is depicted in Fig. X.39.

Amplitude-frequency modulation in the case of the correlated modulating processes. Let us consider the case of combined amplitude-frequency modulation, when amplitude and frequency of radio signal are modulated by the processes, correlated with each other.

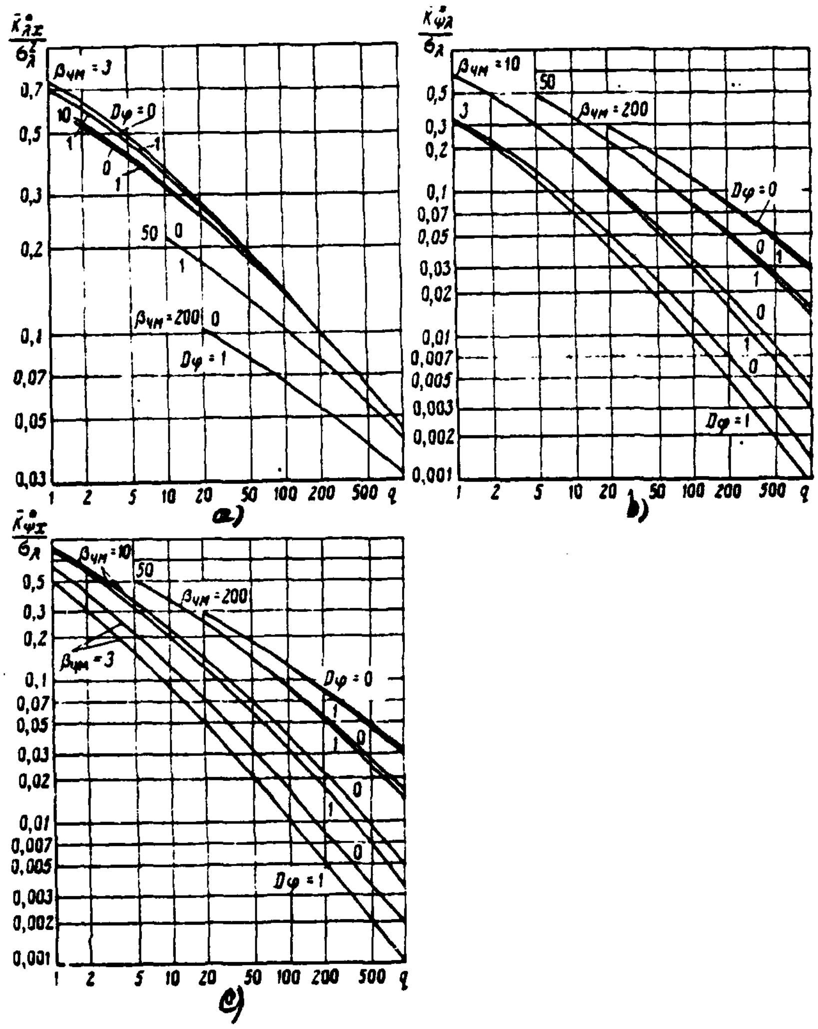



Fig. X.38. Dependence of coefficients $\bar{K}^{\bullet}_{\lambda x}/\sigma^2_{\lambda}$, $\bar{K}^{\bullet}_{\psi\lambda}/\sigma_{\lambda}$ and $\bar{K}^{\bullet}_{\psi x}/\sigma_{\lambda}$ on the signal-to-noise ratio.

Page 339.

Let us record useful radio signal in the form

$$S(t) = M_A \lambda_1(t) \cos[\omega_0 t + \psi(t)];$$
$$\psi(t) = M_\psi \int_0^t \lambda_2(\tau)\, d\tau + \varphi(t). \qquad (X.141)$$

Here $\lambda_1(t)$ and $\lambda_2(t)$ - correlated between themselves random processes, the remaining parameters have previous values.

The behavior of the random parameters $\lambda_1(t)$ and $\lambda_2(t)$ radio signal (X.141) let us assign by the a priori stochastic equations

$$\left.\begin{aligned} \dot{\lambda}_1 &= -\alpha\lambda_1 + n_{\lambda 1}(t); \quad \dot{\psi} = M_\psi \lambda_2 + \dot{\varphi}(t); \\ \dot{\lambda}_2 &= -\beta\lambda_2 + n_{\lambda 2}(t); \quad \dot{\varphi} = n_\varphi(t). \end{aligned}\right\} \qquad (X.142)$$

Processes $\lambda_1(t)$ and $\lambda_2(t)$ can be obtained at the output of the integrating chains/networks RC (Fig. X.40), on inputs of which acts white noise $\zeta(t)$. In this case

$$n_{\lambda 1}(t) = \alpha\zeta(t); \qquad n_{\lambda 2}(t) = \beta\zeta(t).$$

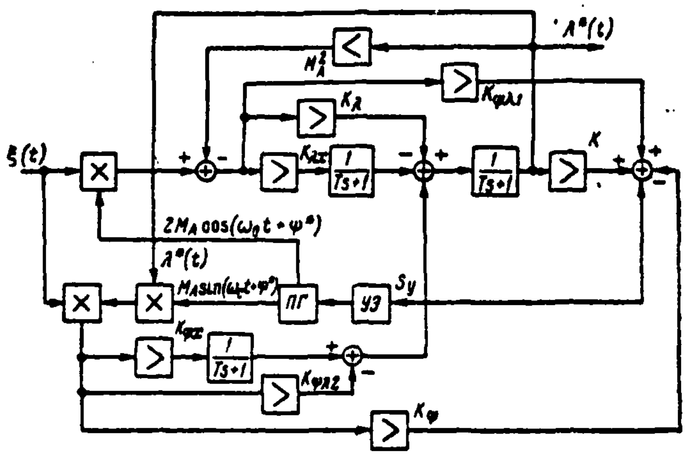



Fig. X.39. Structural scheme of optimum receiver for the radio signal with amplitude-frequency modulation.

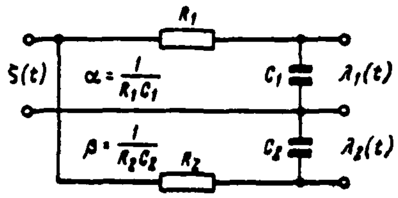



Fig. X.40. Diagram of the formation of informational communications/reports $\lambda_1(t)$ and $\lambda_2(t)$.

Page 340.

The coefficient of the cross correlation between the processes $\lambda_1(t)$ and $\lambda_2(t)$ is equal to

$$R = \frac{2\sqrt{\alpha\beta}}{\alpha+\beta}. \quad \text{(X.143)}$$

In connection with this case the equations of optimum nonlinear filtration take the form

$$\left.\begin{aligned}
&\frac{1}{2}N_{\lambda 1}\, 2\alpha \overline{K}^{\bullet}_{\lambda 1\lambda 1} + (\overline{K}^{\bullet}_{\psi\lambda 1})^2\, \overline{F}_{\psi\psi} + (\overline{K}^{\bullet}_{\lambda 1\lambda 1})^2\, \overline{F}_{\lambda 1\lambda 1} = 0;\\
&\frac{1}{2}N_{\lambda 2} - 2\beta \overline{K}^{\bullet}_{\lambda 2\lambda 2} + (\overline{K}^{\bullet}_{\psi\lambda 2})^2 \overline{F}_{\psi\psi} + (\overline{K}^{\bullet}_{\lambda 1\lambda 2})^2 \overline{F}_{\lambda 1\lambda 1} = 0;\\
&\frac{1}{2}N_{\psi} + 2M_{\psi} \overline{K}^{\bullet}_{\psi\lambda 2} + (\overline{K}^{\bullet}_{\psi\psi})^2\, \overline{F}_{\psi\psi} + (\overline{K}^{\bullet}_{\psi\lambda 1})^2\, \overline{F}_{\lambda 1\lambda 1} = 0;\\
&\frac{1}{2}N_{\lambda 1\lambda 2} - \alpha \overline{K}^{\bullet}_{\lambda 1\lambda 2} - \beta \overline{K}^{\bullet}_{\lambda 1\lambda 2} + \overline{K}^{\bullet}_{\psi\lambda 1}\, \overline{K}^{\bullet}_{\psi\lambda 2}\, \overline{F}_{\psi\psi} + \overline{K}^{\bullet}_{\lambda 1\lambda 1} \times\\
&\qquad \times \overline{K}^{\bullet}_{\lambda 1\lambda 2}\, \overline{F}_{\lambda 1\lambda 1} = 0;\\
&-\alpha\, \overline{K}^{\bullet}_{\psi\lambda 1} + M_{\psi}\, \overline{K}^{\bullet}_{\lambda 1\lambda 2} + \overline{K}^{\bullet}_{\psi\psi}\, \overline{K}^{\bullet}_{\psi\lambda 1}\, \overline{F}_{\psi\psi} + \overline{K}^{\bullet}_{\psi\lambda 1}\, \overline{K}^{\bullet}_{\lambda 1\lambda 1} \times\\
&\qquad \times \overline{F}_{\lambda 1\lambda 1} \quad 0;\\
&\frac{1}{2}N_{\psi\lambda 2} - \beta \overline{K}^{\bullet}_{\psi\lambda 2} + M_{\psi} \overline{K}^{\bullet}_{\lambda 2\lambda 2} + \overline{K}^{\bullet}_{\psi\psi} \overline{K}^{\bullet}_{\psi\lambda 2} \overline{F}_{\psi\psi} + \overline{K}^{\bullet}_{\psi\lambda 1} \overline{K}^{\bullet}_{\lambda 1\lambda 2} \overline{F}_{\lambda 1\lambda 2} = 0.
\end{aligned}\right\} \quad \text{(X.144)}$$

For the transition/junction to the dimensionless quantities let

us divide the first equation by $\sigma_{\lambda 1}^2$, the second - by $\sigma_{\lambda 2}^2$, the fourth - by $\sigma_{\lambda}\sigma_{\lambda 2}$, the fifth - by $\sigma_{\lambda 1}$, the sixth - by $\sigma_{\lambda 2}$; let us introduce signal-to-noise ratio $q = M_{A}^{2}\sigma_{\lambda 1}^{2}/2\alpha N_0$ and the index of frequency modulation $\beta_{\text{ЧМ}} = \sigma_{\omega}/\beta$.

Then taking into account the relationships/ratios

$$\left.\begin{aligned} &\frac{1}{2}N_{\lambda 1} = 2\alpha\sigma_{\lambda 1}^2;\quad \frac{1}{2}N_{\lambda 2} = 2\beta\sigma_{\lambda 2}^2;\\ &\frac{1}{2}N_{\lambda 1\lambda 2} = 2\alpha\sigma_{\lambda 2}^2 = 2\beta\sigma_{\lambda 1}^2;\quad M_{\varphi} = \frac{\sigma_{\omega}}{\sigma_{\lambda 2}};\quad D_{\varphi} = \frac{N_{\varphi}}{2\beta} \end{aligned}\right\}\quad (X.145)$$

system of equations (X.144) can be recorded thus:

$$\left.\begin{aligned} &1 - \delta_{AM}^2 - q\frac{(\overline{K}_{\psi\lambda 1}^{*})^2}{\sigma_{\lambda 1}^2} - q\delta_{AM}^4 = 0;\\ &1 - \delta_{\text{ЧМ}}^2 - \frac{qm(\overline{K}_{\psi\lambda 2}^{*})^2}{\sigma_{\lambda 2}^2} - \frac{qm(\overline{K}_{\lambda 1\lambda 2}^{*})^2}{\sigma_{\lambda 1}^2\sigma_{\lambda 2}^2} = 0;\\ &D_{\varphi} + \frac{2\beta_{\text{ЧМ}}\overline{K}_{\psi\lambda 2}^{*}}{\sigma_{\lambda 2}} - 2qm(\overline{K}_{\psi\psi}^{*})^2 - \frac{2qm(\overline{K}_{\psi\lambda 1}^{*})^2}{\sigma_{\lambda 1}^2} = 0; \end{aligned}\right\}\quad (X.146)$$

$$\left.\begin{aligned} &2\sqrt{m} - \frac{m\overline{K}_{\lambda 1\lambda 2}^{*}}{\sigma_{\lambda 1}\sigma_{\lambda 2}} - \frac{\overline{K}_{\lambda 1\lambda 2}^{*}}{\sigma_{\lambda 1}\sigma_{\lambda 2}} - \frac{2qm\overline{K}_{\psi\lambda 1}^{*}\overline{K}_{\psi\lambda 2}^{*}}{\sigma_{\lambda 1}\sigma_{\lambda 2}} -\\ &\qquad - \frac{2qm\delta_{AM}^2\overline{K}_{\lambda 1\lambda 2}^{*}}{\sigma_{\lambda 1}\sigma_{\lambda 2}} = 0;\\ &\frac{m\overline{K}_{\psi\lambda 1}^{*}}{\sigma_{\lambda 1}} - \frac{\beta_{\text{ЧМ}}\overline{K}_{\lambda 1\lambda 2}^{*}}{\sigma_{\lambda 1}\sigma_{\lambda 2}} - \frac{2qm\overline{K}_{\psi\psi}^{*}\overline{K}_{\psi\lambda 1}^{*}}{\sigma_{\lambda 1}} + \frac{2qm\delta_{AM}^2\overline{K}_{\psi\lambda 1}^{*}}{\sigma_{\lambda 1}} = 0;\\ &\frac{\overline{K}_{\psi\lambda 2}^{*}}{\sigma_{\lambda 2}} - \beta_{\text{ЧМ}}\delta_{\text{ЧМ}}^2 + \frac{2qm\overline{K}_{\psi\psi}^{*}\overline{K}_{\psi\lambda 2}^{*}}{\sigma_{\lambda 2}} + \frac{2qm\overline{K}_{\psi\lambda 1}^{*}\overline{K}_{\lambda 1\lambda 2}^{*}}{\sigma_{\lambda 1}\sigma_{\lambda 2}} = 0. \end{aligned}\right\}\quad (X.146)$$

Page 341.

Here $\delta_{AM}^2 \; \frac{\overline{K}_{\lambda 1\lambda 1}^{*}}{\sigma_{\lambda 1}^2}$ the relative error of the filtration of

communication/report $\lambda_1(t)$; $\delta^2_{qM} = \frac{\overline{K}^*_{\lambda 2 \lambda 2}}{\sigma^2_{\lambda 2}}$ - the relative error of the filtration of communication/report $\lambda_2(t)$;

$$m = \frac{\alpha}{\beta} = \frac{2 - R^2 + 2\sqrt{1 - R^2}}{R^2}. \qquad (X.147)$$

System of equations (X.146) is solved by numerical method on TsVM for series/row of values $\beta_{qM}$, $R$, $q$ and $D_\varphi$. In this case it was assumed that α=const, and the coefficient of correlation R is changed as a result of the change β, moreover 0<β≤α.

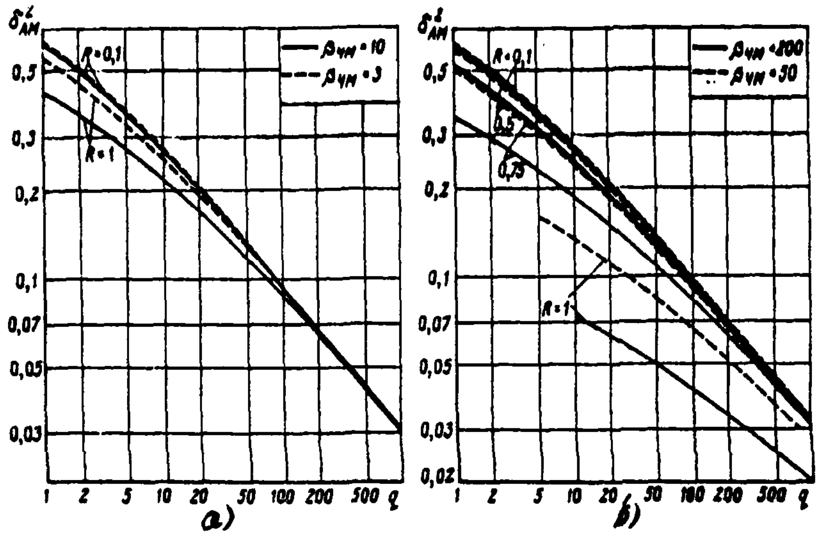



Fig. X.41. Dependence of the square of the relative error of the filtration of communication/report $\lambda_1(t)$ on the signal-to-noise ratio.

Page 342.

Fig. X.41 gives the dependences of the square of the relative error of the filtration of communication/report $\lambda_1(t)$ on the signal-to-noise ratio with the different values of the coefficient of correlation and index of frequency modulation

From the graphs it follows that the decrease of the cross correlation between the processes $\lambda_1(t)$ and $\lambda_2(t)$ leads to an

increase in the relative error of the filtration of process $\lambda_1(t)$. The fluctuations of phase (up to $D_\varphi = 1$) virtually do not affect the relative error of the filtration of communication/report $\lambda_1(t)$.

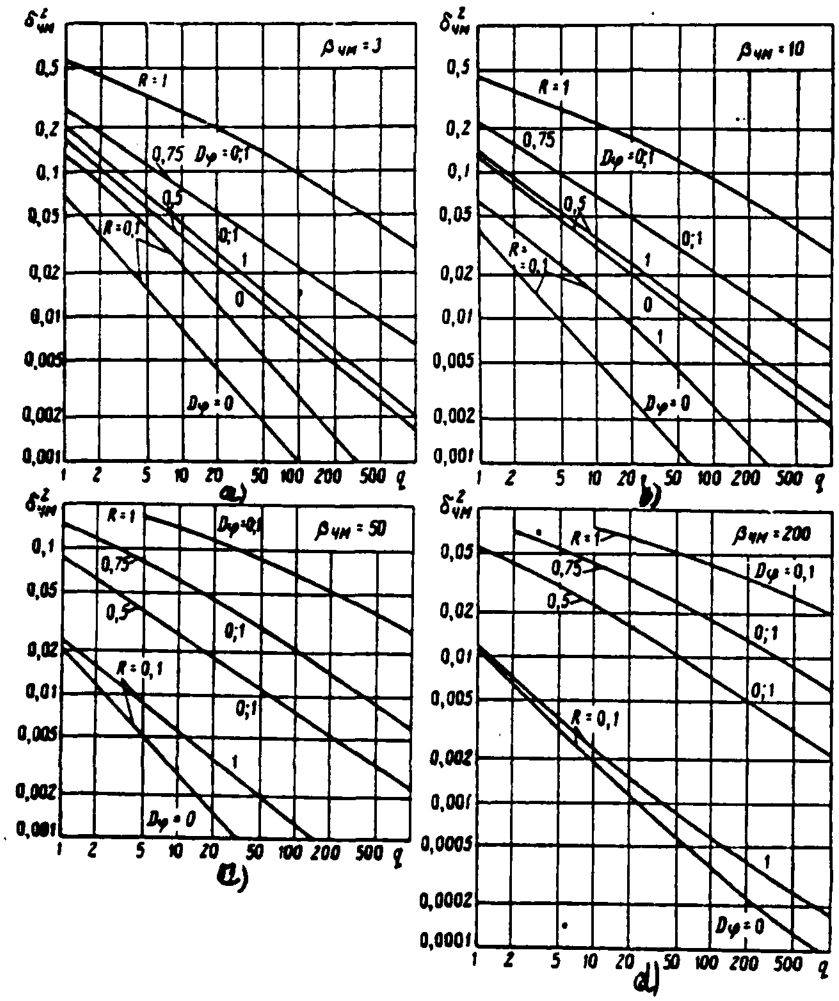



Fig. X.42. Dependence of the square of the relative error of the filtration of communication/report $\lambda_2(t)$ on the signal-to-noise ratio.

Page 343.

Fig. X.42 gives the dependences of the square of the relative

error of the filtration of communication/report $\lambda_2(t)$ on the signal-to-noise ratio with the different values of the coefficient of cross correlation and index of frequency modulation. From the graphs it is evident that the attenuation of correlation between the processes $\lambda_1(t)$ and $\lambda_2(t)$ leads to the decrease of the relative error of the filtration of communication/report $\lambda_2(t)$. This is explained by the fact that with the decrease of the coefficient of cross correlation the signal-to-noise ratio in the channel of the extraction of communication/report $\lambda_2(t)$ increases in m of times in comparison with the signal-to-noise ratio q in the channel of the extraction of communication/report $\lambda_1(t)$. For example, m=4.9 with R=0.75; m=13.9 with R=0.5 and m=398 with R=0.1.

The structural scheme of optimum receiver is analogous to the structural scheme of the optimum sensing transducer AM-ChM, given earlier (see Fig. X.31). Process $\lambda_1(t)$ is selected in the channel of synchronous receiver, and $\lambda_2(t)$ - in the system FAPCh.

5. Optimum reception/procedure of radio signal against the background of white and Markov noises.

Let us consider the case, when oscillation at the input of receiver takes the form

$$\xi(t) = M_A\lambda(t)\cos[\omega_0 t + \varphi(t)] + x(t) + n(t); \qquad \text{(X.148)}$$

here x(t) - normal Markov broadband noise. The remaining parameters of useful signal make the same sense, as for the two-band signal, determined by expression (X.21).

Let the character of a change in the parameters is described by the a priori stochastic differential equations

$$\left.\begin{aligned}\dot{\lambda} &= -\alpha\lambda + n_\lambda(t);\\ \dot{\varphi} &= n_\varphi(t);\\ \dot{x} &= -\nu x + n_x(t),\end{aligned}\right\} \quad (X.149)$$

where $n_\lambda(t)$, $n_\varphi(t)$ and $n_x(t)$ - mutually independent white noises with the zero average/mean values and the autocorrelation functions

$$\left.\begin{aligned}M[n_\lambda(t_1)n_\lambda(t_2)] &= \frac{1}{2}N_\lambda\delta(\tau);\\ M[n_\varphi(t_1)n_\varphi(t_2)] &= \frac{1}{2}N_\varphi\delta(\tau);\\ M[n_x(t_1)n_x(t_2)] &= \frac{1}{2}N_x\delta(\tau),\end{aligned}\right\} \quad (X.150)$$

where $\tau=t_2-t_1$.

Page 344.

In this case the equations of the optimum nonlinear filtration, which determine the structure of optimum receive . take the form

$$\left.\begin{aligned}\dot{\lambda}^* &= -\alpha\lambda^* + \bar{K}^*_{\lambda\lambda}F_\lambda;\\ \dot{\varphi}^* &= \bar{K}^*_{\varphi\varphi}F_\varphi;\\ \dot{x}^* &= -\nu x^* + \bar{K}^*_{xx}F_x;\end{aligned}\right\} \quad (X.151)$$

here

$$\left.\begin{aligned} F_\lambda &= \frac{M_A}{N_0}[2\xi(t)\cos(\omega_0 t+\varphi^*) - M_A\lambda^* - 2x^* \times \\ &\quad\times \cos(\omega_0 t+\varphi^*)]; \\ F_\varphi &= -\frac{1}{N_0}[2\xi(t)\, M_A\lambda^* \sin(\omega_0 t+\varphi^*) - \\ &\quad - 2M_A\lambda^* x^* \sin(\omega_0 t+\varphi^*)]; \\ F_x &= \frac{2}{N_0}[\xi(t) - x^* - M_A\lambda^* \cos(\omega_0 t+\varphi^*)]. \end{aligned}\right\} \quad (X.152)$$

From the solution of the corresponding system of equations we will obtain the following values of cumulants:

$$\left.\begin{aligned} \bar{K}^*_{\lambda\lambda} &= \frac{\alpha N_0}{M_A^2}\left(\sqrt{1+\frac{M_A^2 N_\lambda}{2\alpha^2 N_0}}-1\right); \\ \bar{K}^*_{\varphi\varphi} &= \frac{1}{M_A \sigma_\lambda}\sqrt{\frac{N_0 N_\varphi}{2}}; \\ \bar{K}^*_{xx} &= \frac{\nu N_0}{2}\left(\sqrt{1+\frac{N_x}{\nu^2 N_0}}-1\right). \end{aligned}\right\} \quad (X.153)$$

The square of the relative error of the filtration of communication/report $\lambda(\tau)$ is determined by the formula

$$\delta^2_{AM} = \frac{1}{2q}\left(\sqrt{1+4q}-1\right); \quad q = \frac{M_A^2 \sigma_\lambda^2}{2\alpha N_0}. \qquad (X.154)$$

From the comparison of formulas (X.22) and (X.154) it follows that the presence, at the input of the optimum receiver of a priori known further additive noise x(t), does not worsen/impair the freedom from interference of the radio reception of communication/report λ(t).

Page 345.

After substituting into system (X.151) of the value of derivatives $F_\lambda$, $F_\varphi$, $F_x$, found from formulas (X.152), after simple transformations we will obtain the equations, which determine the structure of the optimum receiver:

$$\left.\begin{aligned}\lambda^* &= \frac{K_\lambda}{T_\lambda s+1}\,[2\xi(t)\cos(\omega_0 t+\varphi^*) - M_A\lambda^* - \\ &\qquad - 2x^*\cos(\omega_0 t+\varphi^*)];\\ \dot{\varphi}^* &= -S_y K_\varphi\,[2\xi(t)\,M_A\lambda^*\sin(\omega_0 t+\varphi^*) - \\ &\qquad - 2M_A\lambda^* x^*\sin(\omega_0 t+\varphi^*)];\\ x^* &= \frac{K_x}{T_x s+1}\,[\xi(t) - x^* - M_A\lambda^*\cos(\omega_0 t+\varphi^*)];\end{aligned}\right\} \quad (X.155)$$

here

$$K_\lambda = \frac{M_A K^*_{\lambda\lambda}}{\alpha N_0};\quad K_\varphi\ \frac{K^*_{\varphi\varphi}}{S_y N_0};\quad K_x = \frac{2K^*_{xx}}{\nu N_0};\quad T_\lambda = \alpha^{-1};\quad T_x = \nu^{-1}.$$

One of the possible versions of the structural scheme of the optimum receiver, which simulates these equations, is depicted in Fig. X.43.

As can be seen from equations (X.155) and Fig. X.43, optimum receiver has two channels. In the first channel the optimum linear filtration of the broadband process of x(t) proceeds from the white noise n(t). In second channel (informational) the quasi-coherent reception/procedure of useful radio signal is realized. Reference signal for the synchronous detector is developed by system FAPCh.

As in the first, that and in second channel is a subtractor. In the first subtractor (Fig. X.44a) from the oscillation $\xi(t)$ accepted is subtracted the "copy" of useful radio signal $M_A\lambda^*(t)\cos[\omega_0 t + \varphi^*(t)]$. In the subtractor of informational channel (Fig. 44b) is compensated the process $2x(t)\cos[\omega_0 t+\varphi^*(t)]$. The compensation will be better, the nearer the evaluation/estimate of the parameters of oscillation $\xi(t)$ to its true values.

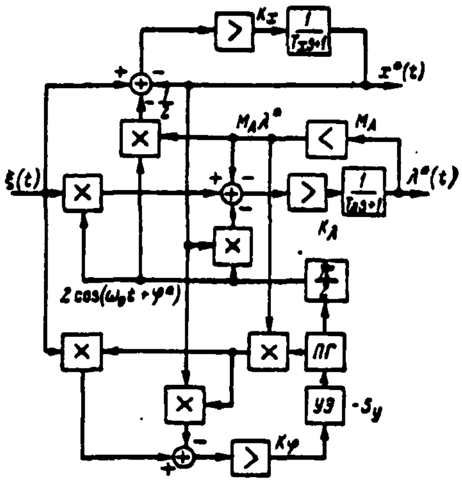



Fig. X.43. Structural scheme of the optimum receiver of two-band radio signal when the white and broadband normal of noises is present.

Page 346.

The latter, obviously, is implemented with the sufficiently high value of signal-to-noise ratio. Analogously occurs the compensation for process $2M_A\lambda^*(t)x^*(t)\cos[\omega_0 t+\varphi^*(t)]$ in the subtractor FAPCh (Fig. X.44c).

Let us consider the optimum reception/procedure of the two-band radio signal, when, besides the white noise, on the input of receiver abnormal (Rayleigh) Markov noise acts. Oscillation at the input of receiver in this case takes the form

$$\xi(t) = M_A \lambda(t) \cos[\omega_0 t + \varphi(t)] + E(t) + n(t); \qquad (X.156)$$

here E(t) - the broadband interference, distributed according to the Rayleigh law. The remaining parameters of oscillation make previous sense.

Let us assign the character of a change in the parameters of oscillation by the following a priori stochastic differential equations:

$$\left.\begin{aligned} \dot{\lambda} &= -\alpha\lambda + n_\lambda(t); \\ \dot{\varphi} &= n_\varphi(t); \\ \dot{E} &= -\nu E + \frac{1}{4E} N_E + n_E(t), \end{aligned}\right\} \qquad (X.157)$$

where the mutually independent white noises have the autocorrelation functions

$$\left.\begin{aligned} M[n_\lambda(t_1) n_\lambda(t_2)] &= \frac{1}{2} N_\lambda \delta(t_2 - t_1); \\ M[n_\varphi(t_1) n_\varphi(t_2)] &= \frac{1}{2} N_\varphi \delta(t_2 - t_1); \\ M[n_E(t_1) n_E(t_2)] &= \frac{1}{2} N_E \delta(t_2 - t_1). \end{aligned}\right| \qquad (X.158)$$

650

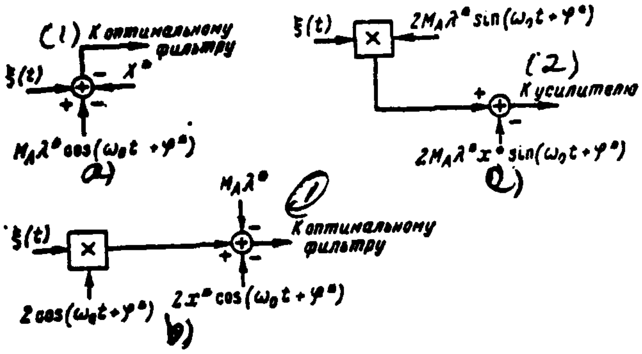



Fig. X.44. Diagram of subtraction device.

Key: (1). To optimum filler. (2). To amplifer.

Page 347.

In connection with the case of the equation of the nonlinear filtration in question, which determine the structure of optimum receiver, they take the form

$$\left.\begin{aligned} \dot{\lambda}^* &= -\alpha\lambda^* + \overline{K}^*_{\lambda\lambda}F_\lambda; \\ \dot{\varphi}^* &= \overline{K}^*_{\varphi\varphi}F_\varphi; \\ \dot{E}^* &= -\nu E^* + \frac{1}{4E^*}N_E + \overline{K}^*_{EE}F_E. \end{aligned}\right\} \qquad (X.159)$$

Here

$$\left.\begin{aligned} F_\lambda &= \frac{M_A}{N_0}\left[2\xi(t)\cos(\omega_0 t+\varphi^*) - M_A\lambda^* - 2E^*\cos(\omega_0 t+\varphi^*)\right]; \\ F_\varphi &= -\frac{1}{N_0}\left[2\xi(t) M_A\lambda^*\sin(\omega_0 t+\varphi^*) - \right. \\ &\quad \left. - 2M_A\lambda^* E^*\sin(\omega_0 t+\varphi^*)\right]; \\ F_E &= \frac{2}{N_0}\left[\xi(t) - E^* - M_A\lambda^*\cos(\omega_0 t+\varphi^*)\right]. \end{aligned}\right\} \quad (X.160)$$

Averaged values of cumulants $\bar{K}^*_{\lambda\lambda}$, $\bar{K}^*_{\varphi\varphi}$, $\bar{K}^*_{EE}$ are determined by the following expressions, found from the solution of the corresponding system of equations:

$$\left.\begin{aligned} \bar{K}^*_{\lambda\lambda} &= \frac{\alpha N_0}{M_A^2}\left(\sqrt{1+\frac{M_A^2 N_\lambda}{2\alpha^2 N_0}} - 1\right); \\ \bar{K}^*_{\varphi\varphi} &= \frac{1}{M_A\sigma_\lambda}\sqrt{\frac{1}{2}N_0 N_\varphi}; \\ \bar{K}^*_{EE} &= \frac{3}{4}\nu N_0\left(\sqrt{1+\frac{4N_E}{9\nu^2 N_0}} - 1\right). \end{aligned}\right\} \quad (X.161)$$

The square of the relative error of the filtration of communication/report λ(t) will be equal to

$$\delta^2_{ДМ} = \frac{1}{2q}\left(\sqrt{1+4q} - 1\right); \quad q = \frac{M_A^2\sigma_\lambda^2}{2\alpha N_0}. \qquad (X.162)$$

From expressions (X.22) and (X.162) it follows that the presence of noise E(t) at the input of optimum receiver does not worsen/impair the freedom from interference of the radio reception of communication/report λ(t).

Page 348.

After substituting into system (X.159) of the value of derivatives $F_\lambda$, $F_\varphi$ and $F_E$, found from formulas (X.160), after simple transformations we will obtain the equations, which determine the structure of the optimum receiver:

$$\left.\begin{aligned}
\lambda^* &= \frac{K_\lambda}{T_\lambda s+1}\,[2\xi(t)\cos(\omega_0 t+\varphi^*) - M_A\lambda^* - \\
&\qquad - 2E^*\cos(\omega_0 t+\varphi^*)]; \\
\dot{\varphi}^* &= -S_y K_\varphi\,[2\xi(t)\,M_A\lambda^*\sin(\omega_0 t+\varphi^*) - \\
&\qquad - 2M_A\lambda^* E^*\sin(\omega_0 t+\varphi^*)]; \\
E^* &= \frac{1}{T_E s+1}\left\{K_E\,[\xi(t) - E^* - M_A\lambda^*\cos(\omega_0 t+\varphi^*)] + \right.\\
&\qquad \left. + \frac{1}{2}\,\frac{\mathrm{M}\,[E^2]}{E^*}\right\}.
\end{aligned}\right\} \quad (X.163)$$

Here

$$K_\lambda = \frac{M_A \overline{K}^*_{\lambda\lambda}}{\alpha N_0};$$

$$K_\varphi = \frac{\overline{K}^*_{\varphi\varphi}}{S_y N_0};$$

$$K_E = \frac{2\overline{K}^*_{EE}}{\nu N_0}.$$

The version of the structural scheme of the optimum receiver, which simulates these equations, is depicted in Fig. X.45.

Optimum receiver has two channels. In the first channel the

optimum linear filtration of process of E(t) from the white noise n(t) is realized. Second channel (informational) implements the quasi-coherent processing of useful radio signal. Reference signal for the synchronous detection is developed by system FAPCh.

In the channels of extraction E*(t) and λ*(t), and also in FAPCh there are subtractors, the principle of work of which is examined above.

6. Optimum reception/procedure of radio signal against the background of white and narrow-band noises.

Let us assume that the oscillation accepted can be recorded in the form

$$\xi(t) = M_A\lambda(t)\cos[\omega_0 t + \varphi(t)] + \zeta(t) + n(t). \qquad (X.164)$$

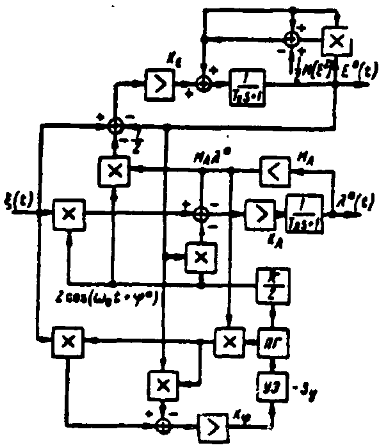


Fig. X.45. Structural scheme of the optimum receiver of two-band radio signal when the white and broadband abnormal Markovian of noises is present.

Page 349.

Here $\xi(t)=E(t)\cos[\omega t+\psi(t)]$ - the narrow-band process, that envelopes which E(t) is distributed according to the law of Rayleigh, and phase $\psi(t)$ - is even to; $\omega$ - a priori known value of medium frequency.

The remaining parameters of oscillation make previous sense.

The character of a change in the parameters of oscillation (X.164) let us assign by the following a priori stochastic equations:

$$\left.\begin{aligned} \dot{\lambda} &= -\alpha\lambda + n_\lambda(t); \\ \dot{\varphi} &= n_\varphi(t); \\ \dot{E} &= -\mu E + \frac{1}{4E} N_E + n_E(t); \\ \dot{\psi} &= n_\psi(t), \end{aligned}\right\} \qquad (X.165)$$

where the autocorrelation functions of mutually independent white noises $n_\lambda(t), n_\varphi(t), n_E(t)$ and $n_\psi(t)$ are respectively equal to:

$$\left.\begin{aligned} M[n_\lambda(t_1) n_\lambda(t_2)] &= \frac{1}{2} N_\lambda \delta(t_2 - t_1); \\ M[n_\varphi(t_1) n_\varphi(t_2)] &= \frac{1}{2} N_\varphi \delta(t_2 - t_1); \\ M[n_E(t_1) n_E(t_2)] &= \frac{1}{2} N_E \delta(t_2 - t_1); \\ M[n_\psi(t_1) n_\psi(t_2)] &= \frac{1}{2} N_\psi \delta(t_2 - t_1). \end{aligned}\right\} \qquad (X.166)$$

In the given case the equations of the nonlinear filtration, which determine the structure of optimum receiver, take the form

$$\left.\begin{aligned} \dot{\lambda}^* &= -\alpha\lambda^* + \overline{K}^*_{\lambda\lambda} F_\lambda; \\ \dot{\varphi}^* &= \overline{K}^*_{\varphi\varphi} F_\varphi; \\ \dot{E}^* &= -\mu E^* + \frac{1}{4E^*} N_E + \overline{K}^*_{EE} F_E; \\ \dot{\psi}^* &= \overline{K}^*_{\psi\psi} F_\psi. \end{aligned}\right\} \qquad (X.167)$$

Here

$$F_\lambda = \frac{M_A}{N_0}\{2\xi(t)\cos(\omega_0 t + \varphi^*) - M_A\lambda^* - E^*\cos[(\omega_0 - \omega)t + (\varphi^* - \psi^*)]\};$$
$$F_\varphi = -\frac{1}{N_0}\{2\xi(t) M_A\lambda^* \sin(\omega_0 t + \varphi^*) - M_A\lambda^* E^* \sin[(\omega_0 - \omega)t + (\varphi^* - \psi^*)]\}; \quad (X.168)$$

$$F_E = \frac{1}{N_0}\{2\xi(t)\cos(\omega t + \psi^*) - E^* - M_A\lambda^*\cos[(\omega_0 - \omega)t + (\varphi^* - \psi^*)]\};$$
$$F_\psi = -\frac{1}{N_0}\{2\xi(t) E^* \sin(\omega t + \psi^*) - M_A\lambda^* E^* \sin[(\omega - \omega_0)t + (\psi^* - \varphi^*)]\}. \quad (X.168)$$

Page 350.

From the solution of the corresponding system of equations we find expressions for the cumulants:

$$\overline{K}^*_{\lambda\lambda} = \frac{\alpha N_0}{M_A^2}\left(\sqrt{1 + \frac{M_A^2 N_\lambda}{2\alpha^2 N_0}} - 1\right);$$
$$\overline{K}^*_{\varphi\varphi} = \frac{1}{M_A\sigma_\lambda}\sqrt{\frac{1}{2}N_0 N_\varphi};$$
$$\overline{K}^*_{EE} = \frac{3}{2}\mu N_0\left(\sqrt{1 + \frac{2N_E}{9\mu^2 N_0}} - 1\right);$$
$$\overline{K}^*_{\psi\psi} = \sqrt{\frac{N_0 N_\psi}{2M[E^2]}}. \quad (X.169)$$

Square of the relative error of the filtration

$$\delta^2_{ДМ} = \frac{1}{2q}\left(\sqrt{1 + 4q} - 1\right); \quad q = \frac{M_A^2\sigma_\lambda^2}{2\alpha N_0}. \quad (X.170)$$

From expressions (X.22) and (X.170) it is evident that the

presence of selective interference does not worsen/impair the freedom from interference of the radio reception of communication/report λ(t).

After substituting into expressions (X.167) of the value of derivatives $F_{\varphi}, F_{\lambda}$, $F_E$ and $F_{\psi}$, determined by formulas (X.168), after transformations we will obtain the equations, which determine the structure of the optimum receiver:

$$\left.\begin{aligned}
\lambda^* &= \frac{K_\lambda}{T_\lambda s+1}\{2\xi(t)\cos(\omega_0 t+\varphi^*) - M_A\lambda^* - \\
&\quad - E^*\cos[(\omega_0-\omega)t+(\varphi^*-\psi^*)]\}; \\
\dot{\varphi} &= -S_y K_\varphi\{2\xi(t) M_A\lambda^*\sin(\omega_0 t+\varphi^*) - \\
&\quad - M_A\lambda^* E^*\sin[(\omega_0-\omega)t+(\varphi^*-\psi^*)]\}; \\
E^* &= \frac{1}{T_E s+1}\{K_E[2\xi(t)\cos(\omega t+\psi^*) - E^* - \\
&\quad - M_A\lambda^*\cos((\omega_0-\omega)t+(\varphi^*-\psi^*))] + M[E^2]/2E^*\}; \\
\dot{\psi}^* &= -S_y K_\psi\{2\xi(t) E^*\sin(\omega t+\psi^*) - \\
&\quad - M_A\lambda^* E^*\sin[(\omega-\omega_0)t+(\psi^*-\varphi^*)]\}.
\end{aligned}\right\} \quad (X.171)$$

Page 351.

Here

$$K_\lambda = \frac{M_A \overline{K}^*_{\lambda\lambda}}{\alpha N_0}; \quad K_\varphi = \frac{\overline{K}^*_{\varphi\varphi}}{S_y N_0}; \quad K_E = \frac{\overline{K}^*_{EE}}{\mu N_0}; \quad K_\psi = \frac{\overline{K}^*_{\psi\psi}}{S_y N_0};$$
$$T_\lambda = \alpha^{-1}; \quad T_E = \mu^{-1}.$$

The version of the structural scheme of the optimum receiver, which simulates these equations, is given in Fig. X.46.

From equations (X.171) and Fig. X.46 it is evident that the optimum receiver has two channels, in which is realized the quasi-coherent processing of useful radio signal and narrow-band noise. Reference signals are formed/shaped with two systems FAPCh.

The oscillation $\xi(t)$ accepted enters four multipliers $P_1$-$P_4$. In the multiplier $P_3$ (synchronous detector) the oscillation $\xi(t)$ is multiplied with the reference signal of system FAPCh (Fig. X.46, X.47a):

$$\eta(t)=2\xi(t)\cos(\omega_0 t+\varphi^*)=[M_A\lambda(t)\cos(\omega_0 t+\varphi)+$$
$$+E(t)\cos(\omega t+\psi)+n(t)]\,2\cos(\omega_0 t+\varphi^*)=M_A\lambda\cos(\varphi-\varphi^*)+$$
$$+E\cos[(\omega_0-\omega)t+(\varphi^*-\psi)]+2n(t)\cos(\omega_0 t+\varphi^*).$$

The synchronous detector follows the subtractor, in which of the process $\eta(t)$ is subtracted oscillation $E^*\cos[(\omega_0-\omega)t+(\varphi^*-\psi^*)]$. When $\varphi\approx\varphi^*$, $\psi\approx\psi^*$ and $E\approx E^*$, which is implemented in the sufficiently large ratio signal/noise, at the output of subtractor is obtained the additive mixture of informational communication/report and white noise:

$$\lambda(t)+2n(t)\cos(\omega_0 t+\varphi^*).$$

Further communication/report $\lambda(t)$ is filtered out from the white noise by optimum linear filter.

Thus occurs the separation/department of the envelope of

interference E(t) from the useful signal (Fig. 47b), which is interference for the narrow-band process of $E(t)\cos(\omega t+\psi)$.

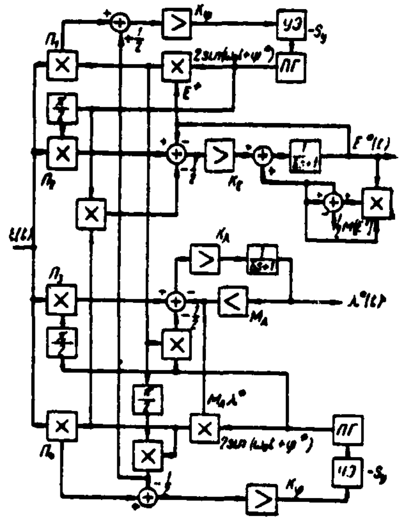


Fig. X.46. Structural scheme of the optimum receiver of two-band radio signal in the presence of white and narrow-band noises.

Page 352.

Analogous processing is implemented in the channels of the synchronization of optimum receiver. The output signals of phase discriminators enter the subtractors, in which occurs the compensation for the processes, which are obtained as a result of the multiplication of "interference" with the reference voltage of the corresponding system FAPCh (Fig. 47c, d).

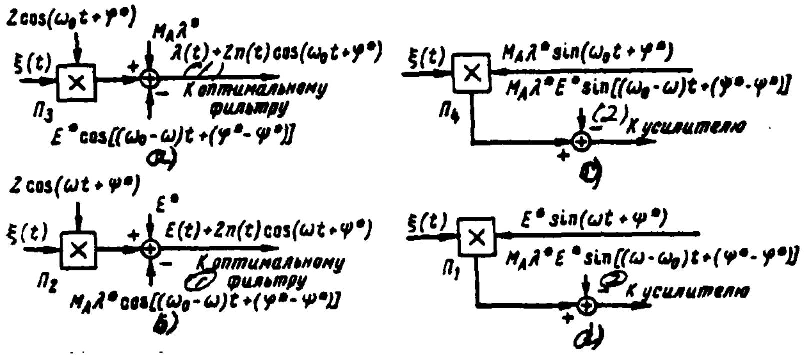



Fig. X.47. Diagram, which elucidates the work of optimum receiver.

Key: (1). To the optimum filter. (2). To amplifier.

Page 353.

Chapter XI.

Special features[redacted] of the [redacted]application of digital computers in statistic studies of nonlinear systems.

1. Prerequisites/premises of use/application of TsVM in statistic studies of nonlinear systems.

In the latter/last 10-15 years the production and the use/application of TsVM acquired enormous spread/scope. The technical and mathematical possibilities of machines substantially were improved, transition/junction from the hand programming to the automatic occurred. Methods and technology of automatic programming were developed actually in the whole scientific and technical direction.

The use/application of algorithmic languages or languages of automatic programming is principal direction in the development of

technology of programming. The principle of the automation of programming in this case consists of the following.

The algorithm of the solution of problem in the machine is written/recorded on the special algorithmic language, simpler for the machines than ordinary formula-word description. This language is constructed according to the simple rules, which do not allow/assume the ambiguous interpretation of one or the other expression. Algorithmic language uses usually a finite number of strictly established/installed symbols, in which are included the letters, digits, signs of mathematical and logical operations, bracket, etc.

Page 354.

The recording of expressions in this language is linear, that excludes such constructions/designs as, for example, two-dimensional tables, fraction, degree, etc.

The algorithm, recorded by programmer with the algorithmic language on the simple sheets of paper or is more frequent on the special forms, they frequently call the program, recorded with the algorithmic language. Further with the help of the keyboard units and the devices/equipment of punching the consecutive mechanical coding of all symbols and their plotting in the coded view of punch carrier

(punch cards or punched tape) is realized.

Prepared thus algorithm is input/embedded through the reader into the machine and with the help of the specific routine, called translator, they process into the ordinary program (in machine instructions and numbers), according to which is implemented the solution of problem.

The algorithmic languages include, for example, the languages of wide designation/purpose for the scientific and engineering calculations ALGOL, FORTRAN languages for the economic calculations COBOL, ALGEC, ALGEM, etc. [2, 12, 17, 29, 38, 41, 42, 57, 68, 87, 98]. There is a considerable number of the so-called problem-oriented languages, specialized for the narrow classes of tasks. The languages named above are not completely general-purpose, but each of them is also calculated for certain circle of tasks.

In the Soviet Union is most widespread the language ALGOL.

Simultaneously with the treatment/processing of algorithm with the help of the translator (this process it is called relaying) occurs the detection of the errors of formal character in the recording of algorithms in the algorithmic language. Information on the discovered errors (character of errors and their place) is put

out to the printing.

The existing translators from ALGOL ensure the production of working program in the codes of instructions of machines with a speed of from several ten to many hundred per minute. This means that to obtaining of program by space, for example, into 3000-4000 instructions is required the time, calculated by minutes or at the worst by several hours.

On the quality the obtained working programs are somewhat inferior to the hand programs: they, as a rule, more space require for the realization of larger time than the programs, comprised by the programmer of high qualification. However, this deficiency/lack is redeemed by the fact that the time, spent by programmer on composition and debugging of programs, during the use of a system of automatic programming is reduced several times. In this case the possibility much of the more careful performance of the algorithm of task appears, which can give the gain, which completely compensates for the losses, as with which it gives the use/application of a translator.

Page 355.

In the presence of the high-quality translator, which ensures

the sufficiently high velocity of relaying, the control/checking of errors into the recording of algorithm and the acceptable quality of programs, series programmers can successfully make programming in the algorithmic language and not deal with the programs in the codes of machine.

Moreover, because of the contemporary methods automatic programming the solution of problems on TsVM became completely available for the wide circle of specialists, who work in different regions and not being professional-programmers.

The known methods of the statistic study of nonlinear systems require the carrying out of the computational work, whose space varies in the extremely wide limits depending on the complexity of the system being investigated and on the formulation of the problem of research. Below for us it is necessary to concern three following classes of the tasks:

The task of calculation we will call obtaining the values, which characterize the properties of system, under its preset structure (i.e., the equations, which describe the operation of system), given values of all parameters of system and its working conditions.

The task of analysis let us name the determination of the

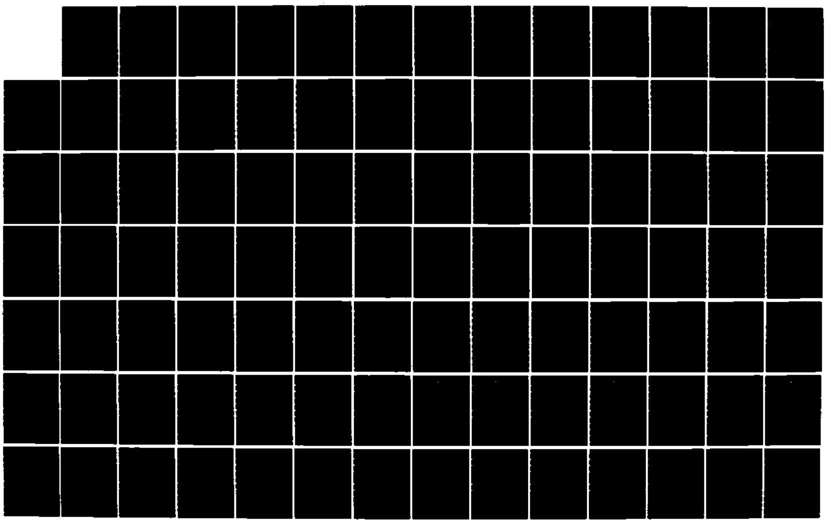
dependence of properties or system characteristics on changes in structure either values of the series/row of the varied parameters of system, working conditions for its, or from any combinations of the factors enumerated above.

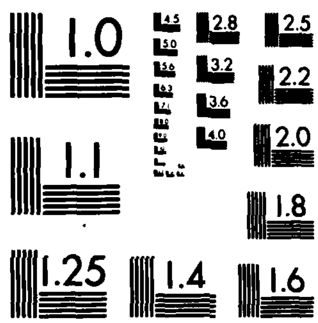
The task of synthesis we will call this selection of the structure of the specific part of system either values of some of its parameters, or that, etc. together, when are ensured properties of system best in a sense under given conditions for its work and given values of those parameters, whose determination was not the target of synthesis.

For the simple systems there are relatively simple analytical methodologies, which make it possible, to solve the problem of calculation, and sometimes also the task of analysis and synthesis with a small volume of the computations, realized with the help of the simplest computational means. With the complication of system the spaces of computations grow. In this case to decide analytically and without the help TsVM of the task of analysis and synthesis proves to be impossible, even if there are analytical methods for the solution of the problem of calculation.

With further complication of systems cease satisfactorily to work the analytical methodologies (usually approximated) of the

AD-A143 297 STATIC METHODS IN THE DESIGN OF NONLINEAR AUTOMATIC 8/9
CONTROL SYSTEMS(U) FOREIGN TECHNOLOGY DIV
WRIGHT-PATTERSON AFB OH N I ANDREYEV ET AL. 27 JUN 84
UNCLASSIFIED FTD-ID(RS)T-1734-83 F/G 12/1 NL






MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

solution of the problem of calculating the system; remain only the methods, which are based on the use/application of computers.

Let us consider the nonlinear system, for which there does not exist the effective methods of the analytical solution of the problem of calculation in the general/common/total setting, formulated above.

Page 356.

The mathematical simulation, which is reduced to the integration with the help of the computers of equations, which describe the dynamics of system, is the fundamental method of the solution of this problem. The solution of the problems of analysis and synthesis can be constructed on the base of the repeated solution of the problem of calculation, i.e., also on the base of mathematical simulation, which will be examined below.

On TsVM it is possible to successfully make the numerical integration of the equations, which describe the operation of system. The results of solution can be obtained with the accuracy, which usually satisfies the requirements, determined by the target of research. However, it must be noted that the machine time, necessary for the numerical integration, for mass TsVM (TsVM of low and average efficiency, which have high speed from hundred to tens of thousands

of operations per second), usually is not so small that it would be possible not to consider it. Therefore appears the problem, which has the large practical importance: to find such numerical methods, which would require during the simulation of automatic control systems on TsVM of the minimum space of computations during the guarantee of their acceptable accuracy.

The problem of analysis is solved by the following path.

The problem of calculation with the sorting/excess of the series/row of the versions of the conditions of task repeatedly is solved. From the result of calculating each version the values of output values are extracted. These values are placed in the conformity with the varied conditions of task. Thus, can be obtained the dependences of values of output values from the varied conditions of task, that also is the target of the solution of the problem of analysis.

In the statistic studies of automatic control systems the most important value has the task of the definition of the characteristics of scattering the output coordinates of system, which carry random ones character as a result of the fact that the system different random interactions (see Chapter I) enter. The study of the dependence of the named characteristics of scattering on changes in

the structure, values of the parameters of system and its working conditions is the development of this task. The second task relates to the analysis, and the first can be attributed both to the tasks of calculation and to the tasks of analysis in the dependence on the setting and methods of its solution. During the use of the methods of Monte Carlo [15, 16] and equivalent disturbances/perturbations (see Chapter IV) this task expediently relating also to the tasks of analysis.

The task of synthesis is the development of the task of analysis, and if during its solution are not used the special analytical methodologies, based not on the simulation of systems (such as, for example, dynamic programming, variational methods, etc.), then the multiple repetition of the task of calculating the system with variations in the conditions of task is also base for the synthesis.

Page 357.

In this case according to the results of each calculation with the help of the exact mathematically formulated algorithm or intuitively is rated/estimated the quality of system, i.e., how good system satisfies the presented to it requirements. As a result of this calculation the version, which has the highest quality, is chosen.

Contemporary TsVM make it possible to successfully automate the process of the solution of the problems of analysis or synthesis. Logical possibilities TsVM more greatly correspond to the character of the solution of these problems, than the tasks of calculation, since here there is a full/total/complete possibility with the help of the mathematical and logical operations to describe the operations, which made men with the results of calculation with the nonautomated methodology of analysis or synthesis, and to realize these operations in the form of machine program. Machine executes these operations considerably more rapid and more precise than man.

Here incomparably more fully/totally/completely are developed the properties of TsVM as the "amplifiers of cognitive abilities" [114], than during the solution of the simple problems of calculation. Full/total/complete automation of the solution of the problems of analysis and synthesis gives not only enormous quantitative effect in the sense of the reduction of time and cost/value of the solution of these problems, but also qualitatively the higher level of the knowledge of reality.

Let us pass to the presentation of the entity of questions of the use/application of TsVM in the statistic studies of nonlinear systems.

2. Principles of simulation on TsVM of automatic control systems.

Let us assume that the nonlinear dynamic system can be described by the normal system of ordinary differential equations of the n order in the vector form

$$\frac{dY}{dx} = F(x, Y), \qquad \text{(XI.1)}$$

where

$$Y = \{y_1, y_2, \ldots, y_n\},\ F = \{f_1, f_2, \ldots, f_n\}.$$

As it was said above, the simulation of dynamic system on TsVM is reduced to the numerical integration of the equations, which describe its dynamics. The numerical methods, used for these purposes, give the possibility to obtain the consecutive values of all variables of system of equations, which follow through the specific steps/pitches or the intervals of the independent variable X(time).

The methods of the numerical integration, which use a constant value of the step/pitch of integration h, are most widely used.

So-called difference methods [11], where values $Y_m$ of the unknown vector Y for $x_m = x_0 + mh$ are determined through consecutive values $F_\kappa$ (for $x = x_\kappa$) the right side of equations (XI.1) and through previous values $Y_{m-j}$ of vector Y, are simplest of such methods.

Page 358.

If are preset values $Y_{m-j}$ and $F_{m-i}$, $i = 0, 1, 2, \ldots, \kappa$, then value can be expressed by the formula $Y_{m+1}$

$$Y_{m+1} = Y_{m-j} + h \sum_{i=0}^{\kappa} \beta_i F_{m-i},$$

where $\beta_i$ - constants, which depend neither on F, nor on m, nor on h.

Frequently replace values $F_{m-i}$ through one of them, for example $F_{m-j}$ and consecutive descending finite differences.

Then we have

$$Y_{m+1} = Y_{m-j} + h \sum_{i=0}^{\kappa} \gamma_i \nabla^i F_{m-j}, \qquad \text{(XI.2)}$$

where

$$\left.\begin{aligned} \nabla^i F_\nu &= \nabla^{i-1} F_\nu - \nabla^{i-1} F_{\nu-1}; \\ \nabla^0 F_\nu &= F_\nu. \end{aligned}\right\} \qquad \text{(XI.3)}$$

If j=0, then we will obtain the formulas of Adams's method:

$$Y_{m+1} = Y_m + h\Big(F_m + \frac{1}{2}\nabla F_m + \frac{5}{12}\nabla^2 F_m + \frac{3}{8}\nabla^3 F_m + \\ + \frac{251}{720}\nabla^4 F_m + \ldots\Big). \qquad \text{(XI.4)}$$

If in equation (XI.4) in the brackets one member $F_m$, there remains only then is obtained Euler's method:

$$Y_{m+1} = Y_m + hF_m. \qquad \text{(XI.5)}$$

Difference methods of numerical integration [11, 40, 62] at present sufficiently are widely worked out. From the nondifference methods the Runge-Kutta method is most widely known. Let us give formulas for the most propagated variety of Runge-Kutta method:

$$\left.\begin{aligned} Y_{m+1} &= Y_m + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4);\\ K_1 &= hF_m;\\ K_2 &= hF\left(x_m + \frac{h}{2},\ Y_m + \frac{K_1}{2}\right);\\ K_3 &= hF\left(x_m + \frac{h}{2},\ Y_m + \frac{K_2}{2}\right);\\ K_4 &= hF(x_m + h,\ Y_m + K_3). \end{aligned}\right\} \qquad \text{(XI.6)}$$

Page 359.

Numerical integration gives the solution of the reference system of differential equations with some errors, which are composed of following components: the errors, caused by an inaccuracy in the assignment of initial conditions; the error of the computation of the right sides of the equations; round-off error during computation $Y_m$ for the formulas of numerical integration; the systematic errors of the formulas of the numerical integration, which consist in the fact that the integral operator in this case is replaced by the approximate operator of the solution of certain finite-difference

equation.

The theory of the accuracy of the numerical integration of ordinary differential equations sufficiently detailed in works [9, 11, 13, 80, 113], etc.

One of the methodologies of the evaluation/estimate of resultant error in the numerical integration of ordinary differential equations was given to M. R. Wurol-Burol [113]. Analogous evaluations/estimates for Euler's method are given in work [34]. Works [11, 40, 62] give the evaluations/estimates of systematic errors.

The errors of approximate solutions, obtained by numerical methods, depend substantially on the properties of the most system being simulated, mainly from its stability (singular points here are not considered). For the stable systems of error from the assignment of initial conditions with the unlimited increase t they vanish, and remaining errors under the limited external influences are also limited. The errors of numerical integration for the unstable systems much more can be unlimited.

The values of errors depend substantially on the step/pitch of numerical integration.

Let us consider the case, when the system being simulated is stable, for obtaining its solutions on segment from 0 to T is applied the sufficiently good method of numerical integration, and accumulated errors of all variables and their components are limited.

Let us designate accumulated error of the calculation of the values of vector Y(t) through $\varepsilon(t)$, and its components, caused by the errors in determination of increments in components Y(t) as a result of an inaccuracy in the calculation of right sides and roundings, and also systematic assumptions, will designate respectively through $\varepsilon_f(t)$ and $\varepsilon_M(t)$, moreover

$$\varepsilon(t) \quad \varepsilon_f(t)+\varepsilon_M(t).$$

Let us assign certain measure for any of these errors, which will be functional from the error as the vector function of time t, which takes positive values, if error is not equal identical to zero, and zero value, if error is identically equal to zero in the segment from 0 to T. Let the measures of errors be respectively equal to $\Delta, \Delta_f$ and $\Delta_M$. These measures will depend on the step/pitch of integration.

The measure of error $\Delta_f$ with the decrease of step/pitch as this is shown in work [113], it grows, moreover with the tendency of step/pitch h toward zero this increase is not limited.

Page 360.

However, the measure of systematic error $\Delta_M$ with the decrease of step/pitch will be reduced. the measure of accumulated error will have a minimum with certain value of step/pitch $h_{opt}$. For $h < h_{opt}$ with the decrease of step/pitch the measure of accumulated error will grow, since the error, caused by an inaccuracy in the calculation of increases in value Y(t), here plays principal role. When $h > h_{opt}$ with an increase in the step/pitch the measure of accumulated error will grow approximately so, as the measure of systematic error, which in this case is principal component, grows.

Taking into account that the labor expense for calculation is inversely proportional to the value of step/pitch h, should be chosen values of h larger than $h_{opt}$, but such, with which is ensured the acceptable accuracy of the numerical integration of the differential equations, .which describe the dynamics of nonlinear system.

During this selection h the accuracy of solution will be determined by predominantly systematic errors. Therefore very important is the study of these errors and the development of such methods of calculations of the output coordinates of the systems,

which ensure the low value of these errors or with the preset accuracy allow/assume steep pitch.

The general/common/total methodologies, worked out for for the evaluation/estimate of the errors of integration, including the mentioned above methodology of M. R. Wurol-Burol [113], qualitatively correctly reflect the effect of the fundamental factors, which call errors, but they quantitatively give the estimate of the magnitude of error with the large reserve. Therefore the accuracy of the solutions of systems of equations during the numerical integration in practice is rated/estimated usually by other methods. One of them consists of the following.

For a small number of versions of initial data the integration with different, gradually decreased values of step/pitch is made. In this case is obtained the sequence of the solutions of system, which virtually converges to certain limit function for each version. The obtained sequence gives the possibility to approximately obtain the limiting value of solution, which is accepted for the true, and to rate/estimate both the its accuracy and accuracy of the solutions, obtained for the different values of step/pitch, and to also select such maximally possible value of the step/pitch of integration, which ensures the required accuracy.

However, this research yet does not give indications of what calculation methods it is necessary to use and, what measures can with one and the same step/pitch of integration lead to the decrease of systematic errors. Response/answer to this question partially can give the general/common/total research of the systematic error, for example, given in work [6]. But high value has also a research of the accuracy of the solutions of the single groups of equations, which considers their specific properties.

Page 361.

In other words, it is very important to investigate the accuracy of the digital models of the single components/links of system and to work out such methods of solution and their such digital models, which would have high accuracy and small labor expense for calculation for these components/links.

3. Digital models of the single components/links of automatic control systems (SAR).

A question about the accuracy of the digital models of the standard components/links SAR, described by ordinary differential equations with the constant coefficients, is sufficiently well studied.

Let us consider this based on the example of the use/application of a finite-difference method of Adams for obtaining the interaction on the output of stable component/link with the fractional rational transfer function

$$W(s) = \frac{b_0 + b_1 s + \ldots + b_m s^m}{a_0 + a_1 s + \ldots + a_n s^n}, \quad m \leqslant n. \qquad \text{(XI.7)}$$

The connection/communication between input x(t) and output y(t) will be determined by the differential equation, which describes the dynamics of the component/link:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f(t); \qquad \text{(XI.8)}$$
$$f(t) = b_m x^{(m)} + b_{m-1} x^{(m-1)} + \ldots + b_1 x' + b_0 x.$$

As is known, the general solution of equation (XI.8) is equal to the sum of its particular solution and general solution of the corresponding homogeneous differential equation with f(t)=0. Let us consider these solutions from the point of view of the systematic errors, which appear during the use of methods of numerical integration, separately.

Let us consider the first particular solution of nonhomogeneous equation (XI.8).

Assuming that x(t) and y(t) satisfy Dirichlet conditions and they can be represented by integral or Fourier series and taking into account the properties, which escape/ensue from the linearity of equation, we investigate the effect of the method of numerical integration only for the case, when

$$x = x_0 e^{st},$$

where $x_0$ and s - some, generally speaking, complex numbers.

Let us consider the use/application of Adams's method for integrating equation (XI.8). In the general case of x(t) it is the arbitrary function, obtained, in particular, in the process of the numerical integration of the differential equations, which describe some another component/link of the automatic control system being investigated.

Page 362.

In this case for integrating equation (XI.8) this organization of the computations, when it is not necessary to resort to the numerical differentiation the variable x(t), is applied, but are used only integration and ordinary arithmetic operations. This does not cause special difficulties; therefore here this question in detail will not be examined.

Let us assume that in the case in question, when $x(t) = x_0 e^{st}$, differentiation x(t) also is not applied, but is used only integration. It is possible to show that during the numerical integration of equation (XI.8) in this manner with the use/application of Adams's method occurs the particular solution

$$y = y_0 e^{st}, \tag{XI.9}$$

where

$$y_0 = \frac{\sum_{i=0}^{m} b_i q^i}{\sum_{j=0}^{n} a_j q^j} x_0; \tag{XI.10}$$

$$q = \frac{e^{sh} - 1}{h\left[1 + \frac{1}{2}(1 - e^{-sh}) + \frac{5}{12}(1 - e^{-sh})^2 + \dots\right]}. \tag{XI.11}$$

During the exact integration

$$y_0 = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{j=0}^{n} a_j s^j} x_0. \tag{XI.12}$$

The distortion of the particular solution of equation (XI.8) in question consists in the fact that value $y_0$, which can be named "complex amplitude", is calculated not from formula (XI.12), which corresponds to exact solution, but from expression (XI.10), analogous to expression (XI.12), where instead of s value q, connected with s with relationship/ratio (XI.11), is substituted.

It is easy to note that

$$\lim_{|sh| \to 0} \frac{q}{s} \quad 1. \qquad \text{(XI.13)}$$

Hence it follows that within the limit, with the infinite decrease of step/pitch, the numerical solution, obtained during the use/application of Adams's method, approaches exact. It is obvious that for obtaining the sufficiently exact solution it is necessary that the value $|sh|$ would be considerably less than 1.

For the case, when s is imaginary, which corresponds to harmonic oscillations at the input of this component/link, latter/last conclusion/output can be formulated thus: for obtaining with a sufficient accuracy of the reaction of the component/link, described by transfer function (XI.7), to the sinusoidal input effect it is necessary that step/pitch value would comprise a small fraction of a period of sinusoid at the input of component/link.

Page 363.

The obtained relationships/ratios easily can be subjected to numerical analysis, and conclusions/outputs can be defined concretely for the single standard components/links - inertial, oscillatory, etc., and also for the formulas of integration with an accuracy to differences in the first, the second and so forth of orders.

Further let us consider what distortions undergo the solutions of the homogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + c_0 y = 0. \qquad \text{(XI.14)}$$

We will assume that for integrating this equation Adams's method is applied also. We will find out solutions in this case in the form of exponential curves $y = Ce^{\mu t}$. For this it is necessary that the equality

$$Ce^{\mu t} \sum_{j=0}^{n} c_j \lambda^j = 0, \qquad \text{(XI.15)}$$

where

$$\lambda = \frac{e^{\mu h}}{\left[h\, 1 + \frac{1}{2}(1 - e^{-\mu h}) + \frac{5}{12}(1 - e^{-\mu h})^2 + \ldots\right]}. \qquad \text{(XI.16)}$$

would occur.

So that equation (XI.14) would have nontrivial ones, different from zero ones solutions of form $y = Ce^{\mu t}$, it is necessary that λ would be the root of the characteristic equation

$$\sum_{j=0}^{n} c_j \lambda^j = 0. \qquad \text{(XI.17)}$$

This equation has n roots, which leads to n linearly independent

approximate particular solutions of equation (XI.14), obtained by Adams's method. In this case the general solution is equal to the linear combination of these particular solutions.

The distortions of the named particular solutions of equation (XI.14) can be easily determined. For example, for the single root $\lambda$ of characteristic equation (XI.17) exact particular solution takes form $y = Ce^{\lambda t}$, and the corresponding solution of the finite-difference equation, which is obtained during the use/application of Adams's method to equation (XI.14), takes form $y = Ce^{\mu t}$. Values $\mu$ and $\lambda$ are connected with relationship/ratio (XI.16). Easily can be obtained the distortions of the particular solutions of equation (XI.14), which correspond to the multiple and complex roots of characteristic equation.

Page 364.

The distortions of particular solutions are linear components of the total systematic error of general solution of the homogeneous equations (XI.14).

It is easy to demonstrate that

$$\lim_{|\lambda h| \to 0} \frac{\mu}{\lambda} = 1.$$

Thus, so that Adams's method would give sufficiently exact solutions, it is necessary that $\lambda h$ it would be sufficiently little in the absolute value.

It is obvious that by condition it is sufficient exact solution of the differential equation, which describes certain component of automatic control system, is the smallness of the step/pitch of integration in comparison with the constant values of the time of this component/link.

Relationships/ratios presented above make it possible to study in detail the characteristics of the digital models of the linear components/links SAR, described by transfer function (XI.7) with different methods of numerical integration, as a result of which it is possible to obtain more concrete/more specific/more actual conclusions. The accuracy of the digital models of linear components/links SAR is sufficiently widely investigated [53, 80, 84, 111]. The results of these research, in particular, confirm that the more exact methods of numerical integration allow/assume larger step/pitch.

Of that presented it above follows that the step/pitch of integration, which ensures the acceptable accuracy of the simulation of entire system as a whole, must be considerably less than the

smallest time constant in the denominators of the transfer functions of the linear stationary components/links of system. Therefore it can seem that for the integration the very low pitch will be required, which leads to the undesirable expenditures of machine time. The task of the searching of the methods of decreasing these expenditures appears.

There are following methods of solution of this task.

The differential equations of the components/links, which have fast time constants, it is possible to integrate with the low pitch, and remaining equations to integrate with the steep pitch. The smoothly changing coefficients of differential equations can be calculated with the steep pitch on the time, and the intermediate values of coefficients can be found via interpolation or extrapolation.

It is possible to apply the methods of numerical integration with the automatic selection of the values of step/pitch, for example, the method of Merson [58]. However, the use of this method it is conjugated/combined with the definite difficulties in the presence in system of delay links and nonlinearity.

Finally, it is possible to use numerical-analytical method for

obtaining the values of output variable/alternating stationary components/links with the fast time constants. It is presented briefly the entity of this method.

Page 365.

Method is applicable when the input interaction x(t) is changed sufficiently smoothly. The permissible value of step/pitch with this method does not depend on the time constants of components/links, but it is determined only by the accuracy of the approximation of the input interaction by the polynomial of the selected degree.

Let us consider the component/link with transfer function (XI.7), described by differential equation (XI.8).

For the homogeneous equation

$$\sum_{j=0}^{n} a_j \frac{d^j y}{dt^j} = 0,$$

corresponding to nonhomogeneous equation (XI.8), can be determined the fundamental system of particular solutions, which consists of the functions of form $t^q e^{\lambda_p t}$, where $\lambda_p$ - roots (generally speaking, complex) the characteristic equation

$$\sum_{j=0}^{n} a_j \lambda^j \quad 0.$$

The general solution of equation (XI.8) can be obtained in the form

$$y = \sum_{r=1}^{S} y_r;$$

$$y_r = e^{\lambda_r t} \sum_{\kappa=0}^{S_r} C_{r\kappa} t^{\kappa} + \int_0^t \left[ \sum_{\kappa=0}^{S_r} B_{r\kappa} (t-u)^{\kappa} e^{\lambda_r (t-u)} \right] f(u)\, du; \quad \text{(XI.18)}$$

$$\sum_{r=1}^{S} S_r = n.$$

Here $S_r$ - multiplicity of root $\lambda_r$ of characteristic equation (XI.17);

$C_{r\kappa}$ - arbitrary constants;

$B_{r\kappa}$ - coefficients, which depend on the roots of characteristic equation;

f(t) - the right side of equation (XI.8);

$$f(t) = \sum_{l=1}^{m} b_l \frac{d^l x}{dt^l}. \quad \text{(XI.19)}$$

Substituting in expression (XI.18) for f(u) its value in accordance with formula (XI.19) and applying integration in parts, we are convinced, that as a result the expression for $y_r$ through x(u),

analogous to equation (XI.18), is obtained but with the changed values of constants.



Therefore we will continue the examination of expressions (XI.18), assuming/setting in them by f(u)=x(u).

Let us introduce the designations

$$y_{r\kappa,q,m}=C_{r\kappa}t^q e^{\lambda_r t_m}+B_{r\kappa}\int_0^{t_m}(t_m-u)^q e^{\lambda_r(t_m-u)}x(u)\,du;$$

$$h=t_m-t_{m-1};\ y_{rm}=y_r(t_m);\ y_m=y(t_m).$$

Then as a result of simple transformations we obtain

$$y_m=\sum_{r=1}^{S}y_{rm};\ y_{rm}=\sum_{\kappa=0}^{S_r}y_{r\kappa,\kappa,m};$$

$$y_{r\kappa,q,m}=e^{\lambda_r h}\sum_{l=0}^{q}C_q^l y_{r\kappa,q-l,m-1}h^l+B_{r\kappa}\int_0^h v^q e^{\lambda_r v}x(t_m-v)\,dv.$$

Final formulas for the numerical integration can be obtained after the replacement of function $x(t_m-v)$ by the polynomial, which approximates it. After this integral, into integrand of which enters function $x(t_m-v)$, easily it is calculated:

$$\int_0^h v^q e^{\lambda_r v}x(t_m-v)\,dv\ \ \sum_{j=0}^{l}p_j x_{m-j},\qquad (XI.20)$$

where $p_j$ - numbers, which depend only on h, q, $\lambda_r$, $l$.

Instead of consecutive values $x_{m-j}$ it is possible to use the descending or ascending differences. For the descending differences we will obtain

$$\int_0^h v^q e^{\lambda_r v} x(t_m - v)\,dv = \sum_{j=0}^{l} q_j \nabla^j x_{m-1},$$

where $q_j$ - also the numbers, which depend on h, q, $\lambda_r$, $l$.

The general/common/total idea of the construction of the algorithm of the calculation of consecutive values $y_m$. is such. Concrete/specific/actual formula dependences for different types of components/links with the real coefficients are given below. The approximating polynomial of the 2nd degree of the form

$$x(t_m - v) = x_{m-1} + \left(1 - \frac{v}{h}\right)\nabla x_{m-1} + \frac{\left(1 - \frac{v}{h}\right)\left(2 - \frac{v}{h}\right)}{2}\nabla^2 x_{m-1}.$$

is selected. First-order components/links

$$W(s) = \frac{b_0 + b_1 s}{a_0 + a_1 s};$$

$$a_0 = 1; \quad a_1 \neq 0; \quad \lambda = \frac{-1}{a_1}; \quad \alpha = \lambda h.$$

Page 367.

1. The calculation made one time before beginning integration:

$$\left.\begin{array}{c}
\alpha = \lambda h;\quad \beta = e^{\alpha};\quad \gamma = b_1\lambda;\quad \delta = \alpha\,(b_0 + \gamma);\\
\theta_0 = \dfrac{e^{\alpha} - 1}{\alpha};\quad \theta_1 = \dfrac{e^{\alpha} - 1 - \alpha}{\alpha^2};\\
\theta_2 = \dfrac{\dfrac{\alpha}{2}\,(e^{\alpha} - 1) + e^{\alpha} - 1 - \alpha - \alpha^2}{\alpha^3};\\
W_0 = y_0 + \gamma x_0.
\end{array}\right\} \qquad (XI.21)$$

2. Calculation made at each step/pitch:

$$y_m = w_m - \gamma x_m;$$
$$w_m = \beta w_{m-1} + \delta\,(\theta_0 x_{m-1} + \theta_1 \nabla x_{m-1} + \theta_2 \nabla^2 x_{m-1}).$$

Components/links of the second order

$$W(s) = \frac{b_0 + b_1 s + b_2 s^2}{a_0 + a_1 s + a_2 s^2},\quad a_2 \neq 0;$$

$\lambda_1$, $\lambda_2$ - the roots of characteristic equation;

$$a_2\lambda^2 + a_1\lambda + a_0 = 0.$$

The 1st case: $a_0 = 1$; $a_1^2 - 4a_2 > 0$. Are given $y_0 = y'(0)$ and $y'_0 = y'(0)$.

Page 368.

1. Calculation, made one time before beginning integration:

$$\left.\begin{aligned}
&\alpha_1 = \lambda_1 h; \quad \alpha_2 = \lambda_2 h;\\
&\beta_1 = e^{\alpha_1}; \quad \beta_2 = e^{\alpha_2};\\
&\theta_{10} = \frac{e^{\alpha_1}-1}{\alpha_1}; \quad \theta_{20} = \frac{e^{\alpha_2}-1}{\alpha_2};\\
&\theta_{11} = \frac{e^{\alpha_1}-1-\alpha_1}{\alpha_1^2}; \quad \theta_{21} = \frac{e^{\alpha_2}-1-\alpha_2}{\alpha_2^2};\\
&\theta_{12} = \frac{\frac{\alpha_1}{2}(e^{\alpha_1}-1)+e^{\alpha_1}-1-\alpha_1-\alpha_1^2}{\alpha_1^3};\\
&\theta_{22} = \frac{\frac{\alpha_2}{2}(e^{\alpha_2}-1)+e^{\alpha_2}-1-\alpha_2-\alpha_2^2}{\alpha_2^3};
\end{aligned}\right\} \quad \text{(XI.22)}$$

$$\left.\begin{aligned}
&p_1 = \frac{b_0+b_1\lambda_1+b_2\lambda_1^2}{a_2(\lambda_1-\lambda_2)}; \quad p_2 = \frac{b_0+b_1\lambda_2+b_2\lambda_2^2}{a_2(\lambda_2-\lambda_1)};\\
&y_{1,0} = \frac{a_2(y_0'-\lambda_2 y_0)+(b_1-b_2\lambda_1)x_0+b_2x_0'}{a_2(\lambda_1-\lambda_2)};\\
&y_{2,0} = \frac{a_2(y_0'-\lambda_1 y_0)+(b_1-b_2\lambda_2)x_0+b_2x_0'}{a_2(\lambda_2-\lambda_1)}.
\end{aligned}\right\} \quad \text{(XI.22)}$$

2. Calculation made at each step/pitch:

$$\left.\begin{aligned} y_m &= \frac{b_2}{a_2} x_m + y_{1,m} + y_{2,m}; \\ y_{1,m} &= \beta_1 y_{1,m-1} + p_1 h(\theta_{10} x_{m-1} + \theta_{11}\nabla x_{m-1} + \theta_{12}\nabla^2 x_{m-1}); \\ y_{2,m} &= \beta_2 y_{2,m-1} + p_2 h(\theta_{20} x_{m-1} + \theta_{21}\nabla x_{m-1} + \theta_{22}\nabla^2 x_{m-1}). \end{aligned}\right\} \quad (XI.23)$$

$$\text{2-й случай:}^{(1)}\ a_0 = 1; \quad a_1^2 - 4a_2 = 0;$$

$$\lambda_1 = \lambda_2 = \lambda = -\frac{a_1}{2a_2} = -\frac{2}{a_1}; \quad a_2 = \frac{1}{\lambda^2}; \quad \alpha = \lambda h.$$

Key: (1) 2nd case.

1. Cacluclation made once before the beginning of integration

$$r_1 = \lambda^2 (b_1 + 2b_2\lambda); \quad r_2 = \lambda^2 (b_0 + b_1\lambda + b_2\lambda^2);$$
$$y_{1,0} = y_0 - b_2\lambda^2 x_0;$$
$$S_0 = y_0' - \lambda y_0 - \lambda^2 (b_1 + b_2\lambda)\, x_0.$$

Further are calculated $\beta, \theta_0, \theta_1, \theta_2$ from the formulas of form (XI.21), and also

$$\theta_1^* = \frac{2\theta_1 - 1}{\alpha}; \quad \theta_2^* = \frac{2\alpha\theta_2 - \theta_1 - \frac{\alpha}{2}}{\alpha^2} = \frac{2\theta_2 + \frac{\theta_1^*}{2} - 1}{\alpha}.$$

2. Calculation, made at each step/pitch:

$$y_m = y_{1,m} + b_2\lambda^2 x_m;$$
$$y_{1,m} = \beta y_{1,m-1} + r_1 h(\theta_0 x_{m-1} + \theta_1 \nabla x_{m-1} + \theta_2 \nabla^2 x_{m-1}) +$$
$$+ r_2 h(\theta_1 x_{m-1} + \theta_1^* \nabla x_{m-1} + \theta_2^* \nabla^2 x_{m-1}) + hS_m;$$
$$F_m = \beta S_{m-1} + r_2 h(\theta_0 x_{m-1} + \theta_1 \nabla x_{m-1} + \theta_2 \nabla^2 x_{m-1}).$$

3-й случай: (1) $a_0 = 0;\quad a_1 = 1;\quad \lambda = -\frac{1}{a_2};\quad \alpha = \lambda h.$

Key: (1). the 3rd case.

Page 369.

1. Calculation, made one time before beginning integration:

$$y_{1,0} = a_2 y_0' + (b_1 + b_2\lambda) x_0 + b_2 x_0';$$
$$B = b_0 + b_1\lambda + b_2\lambda^2;$$
$$C = y_0 - a_2 y' - b_1 x_0 - b_2 x_0'.$$

Are calculated $\beta$, $\theta_0, \theta_1, \theta_2$ from formulas (XI.21).

2. Computations at each step/pitch:

$$y_m = y_{1,m} + b_0 y_{2,m} + C - b_2\lambda x_m;$$
$$y_{1,m} = \beta y_{1,m-1} - Bh(\theta_0 x_{m-1} + \theta_1 \nabla x_{m-1} + \theta_2 \nabla^2 x_{m-1});$$
$$y_{2,m} = y_{2,m-1} + h\left(x_{m-1} + \frac{1}{2}\nabla x_{m-1} + \frac{5}{12}\nabla^2 x_{m-1}\right).$$

In all cases is here assumed that the polynomial, which approximates the input interaction x(t), in the beginning of

integration (t=0) is also known and known values not only $x_0$, but also $x_{-1}$ and $x_{-2}$, which is necessary for computing the differences $\nabla x_{m-1}$ and $\nabla^2 x_{m-1}$ with m=1 and 2.

The 4th case: $a_0=1$; $a_1^2-4a_2<0$; $\lambda_1$ and $\lambda_2$ - complex conjugated/combined. For this case calculation formulas are obtained from relationships/ratios (XI.22) (XI.23) with the substitution in them of complex values $\lambda_1$ and $\lambda_2$. In this case the coefficients $\mu$ and C computed previously will be complex, and differences $\lambda_1-\lambda_2$ and $\lambda_2-\lambda_1$ are imaginary. Values $y_{1,m}$ and $y_{2,m}$ computed consecutively/serially will be complex however with real coefficients $a_j$ and $b_i$ under the actual initial conditions and the input influence of value $y_m=y_{1,m}+y_{2,m}$ must be real. Due to errors in the computations this will not follow carefully itself; however, imaginary part $y_m$, the being part of the error of computation $y_m$, will characterize the value of this error. Therefore it is possible, taking as the solution the real part of sum $y_{1,m}+y_{2,m}$, to also calculate imaginary part for the control/checking.

The described numerical-analytical method can be used also for the complex values of coefficients $a_j$ and $b_i$, and also during the integration along the arbitrary straight line on the complex plane of the independent variable.

As far as simulation on TsVM of linear systems with variable/alternating coefficients is concerned, here it is possible to isolate the case of a slow change in the parameters of the components/links, when it is possible to use the methods, which are used for the linear systems with the constant parameters. If the parameters of system vary rapidly, then in this case numerical methods must be chosen on the basis of the concrete/specific/actual structure of system and its special features/peculiarities.

Page 370.

The same can be said also about the simulation of nonlinear systems.

In the majority of the practical cases nonlinear automatic control system with the variable parameters can be represented in the form of the complex of the components/links, for each of which the methods of simulation are sufficiently clear. For example, if system can be represented in the form of the connections of the components/links of linear with the constant parameters, functional nonlinearity, amplifying circuits with the variable amplification factors, then during the simulation of linear stationary components/links it is possible to use with some limitations by recommendations presented above. Functional nonlinearity and variable

coefficients on TsVM are simulated without any difficulties.

The mentioned limitations during the simulation of linear stationary components/links consist of the following. If the signal, which enters into the input of nonlinearity, is changed smoothly and it is approximated well and it is extrapolated with the help of the polynomials, then output signal in a number of cases can be changed abruptly, which destroys the possibility of approximation/approach by its polynomials. In this case deteriorate also the possibilities of calculating the passage of such signals through the linear components/links. It is obvious that begin much more badly "to work" utilized for this purpose methods of the numerical integration, based on the approximation of input signal with polynomials, if we do not take special measures to account for the disruptions of the integrated function and its derivatives. Such measures, obviously, can be worked out and used, but they will lead to the complication of calculations. When calculations become unjustifiably complicated for the numerical integration of the differential equations, which describe the behavior of system, it is possible to use Runge-Kutta method, which is least sensitive to the nonlinearity.

4. Use/application of TsVM for the research of the accuracy of nonlinear systems.

We will examine such nonlinear systems, for which it is impossible to analytically determine errors with the known values of the exciting factors.

For the research of the errors of such nonlinear systems the methods, which require the use/application of computers, are used.

The important group of methods is based on the approximate replacement of the nonlinear elements of system by linear ones with further use/application to such linearized systems of the methods of the theory of linear dynamic systems.

Page 371.

Let us consider certain nonlinear element or system. Let this component/link enter signal x(t), which can appear as scalar, so also vector random or nonrandom function of time.

Can seem that this signal is knowingly in a sense limited by sufficiently narrow limits, for example $|x(t)-x_0(t)|<a$, where $x_0(t)$ - previously known function and a is sufficiently small, or signal x(t) and its first-order derivative it is sufficiently small on the module, etc. In this case it can seem that in the preset narrow range of a possible change in the signal all nonlinear operations above the

signal can be approximately replaced linear, so that errors from this replacement can be disregarded/neglected.

This procedure is called ordinary linearization, and if it proves to be applicable to all nonlinear elements of system, then entire system thus is linearized. For the linear automatic control systems the methods of the study of accuracy are worked out.

The much great difficulties of calculation and research of the accuracy of nonlinear systems appear when the methods of simple linearization prove to be inapplicable that for the real systems it is ordinary. For these systems it is in principle possible to use the methods of the so-called statistical linearization (see Chapter II).

The methods of statistical linearization at present underwent considerable development and were worked out in connection with different settings the tasks of research and different classes of nonlinearity. The statistical linearization of the nonlinear element/cell, which realizes the nonlinear inertia-free transformation of the general view

$$Y(t) = F[X(t)], \qquad (XI.24)$$

where F is the static characteristic of nonlinear element/cell, is one of the central ideas for many tasks; x(t) and Y(t) - scalar

random functions at its input and output.

If system is gotten soaked, then the probabilistic characteristics of signals at the inputs of all nonlinear elements/cells easily are calculated. But if nonlinear elements/cells are included by elastic feedback, then the calculation of these characteristics is complicated.

In work [34] is presented the method, based on the refinement of the coefficients of statistical linearization for the nonlinear elements and values $m_x$ and $\sigma_x$ by the method of successive approximations, with the method of solution of differential equation relative to the mathematical expectations of the unknown functions and computation by the methods of the theory of the linear dynamic systems of their second moments/torques.

Page 372.

Other methods are based on the use/application of Duncan's method [122] for determining the moments/torques of the first and second order of the output coordinates of linear system, which is reduced to the solution of the system of differential equations relative to these moments/torques, comprised on the base of the reference system of the differential equations of linear dynamic

system. This idea is combined with the simultaneous use of coefficients of statistical linearization for the nonlinear elements/cells, entering the system, which leads to the completed solution of the problem of determining the first two moments/torques of the output coordinates of system. This approach is used in works [3, 35], etc.

Thus, the fundamental methodologies of the study of the accuracy of nonlinear automatic control systems are reduced to the numerical integration of the systems of ordinary differential equations with the preset initial ones of condition. The integrated system is obtained from initial equations, which describe the dynamics of the control system. The integration of this system of equations, strictly speaking, is not simulation.

During the research of complex systems statistical linearization as a result of the unwieldiness is inferior the place for the methods, based on the direct integration of the differential equations, which describe nonlinear automatic control system. The first of them should be named the method for statistical testing (Monte Carlo method) [15].

The Monte Carlo method is the set of procedures for the solution of the problems of the varied ones in its mathematical setting, if

these procedures are based, in the first place, during the construction of the realizations of random variables and functions, which possess the necessary probabilistic characteristics, in the second place, on the statistical processing of these realizations for the purpose of the determination of the unknown values. The determination of the accuracy of system by this method is reduced to the following.

Let the system or its model be preset. The operating on the system exciting factors, which are random functions and values, let us designate through $X_s$, s=1, 2, ..., M. Let us designate their set through $\Xi = \{X_s\}$.

Let us consider the course of solution of task.

Certain set of the conditions of task R is recorded. For system $\Xi$ of random functions and values, which present random interactions, are constructed consecutively/serially its realizations

$$\Xi_j = \{x_{sj}\}, \quad j = 1, 2, \ldots, N.$$

For each of the realizations $\Xi_j$ is made the integration of the system of equations, automatic control system describing behavior, as

a result of which are obtained the values of output variables $y_{i,j}$, i=1, 2, ..., m.

Page 373.

The characteristics of accuracy for this set of the conditions of task R are obtained as a result of statistical processing of the values of variables $y_{i,j}$.

Computations are made for the varied conditions of task R, as a result of which the evaluations of the probabilistic characteristics of output variables, which consider a change in the conditions of task, are obtained.

Statistical processing usually consists in the calculation of the moments/torques of the first and second orders; more rarely we calculate the moments/torques of higher order and the distribution laws.

For example, the evaluations/estimates of mathematical expectation $m_y^*$ and dispersion $D_y^*$ of value y are designed from the formulas of the following form (for simplicity of recording is here omitted index i):

$$m_y^* = \frac{1}{N}\sum_{j=1}^{N} y_j; \quad D_y^* = \frac{1}{N-1}\sum_{j=1}^{N}(y_j - m_y^*)^2.$$

The accuracy of the evaluations/estimates of mathematical expectation $m_y^*$ and dispersion $D_y^*$ can be described by their dispersions

$$D[m_y^*] = \frac{1}{N(N-1)}\sum_{j=1}^{N}(y_j - m_y^*)^2 = \frac{D_y^*}{N};$$

$$\sigma_{m_y^*} = \sqrt{\frac{D_y^*}{N}} = \frac{\sigma_y^*}{\sqrt{N}}; \quad D[D_y^*] = \frac{2D_y^{*2}}{N-1}.$$

Table XI.1 gives the expressed in the percentages values of values,

$$\frac{\sigma_{m_y^*}}{\sigma_y^*} = \frac{1}{\sqrt{N}}; \quad \frac{\sigma_{D_y^*}}{D_y^*} = \sqrt{\frac{2}{N-1}},$$

the characterizing accuracy evaluations/estimates of mathematical expectation and dispersion of value y, i.e., the accuracy of the Monte Carlo method.

The methods of statistical processing and evaluation of accuracy sufficiently detailed [18, 48, 99], etc.

Usually a number of realizations minimally acceptable from accuracy conditions will be N=30-40, which gives the evaluation/estimate of mathematical expectation with average/mean quadratic error 15-20% of $\sigma_{\nu}^{*}$. To more preferably have a number of realizations $N \approx 10^{2}$, which will give for average/mean quadratic errors $\sigma_{m_{\nu}^{*}}$ accuracy is higher than 10%.

The method of the equivalent disturbances/perturbations (see Chapter IV) is the second method of calculation of accuracy, based on the straight/direct integration of system of equations.

Page 374.

The evaluation of the accuracy of result is complicated problem in the method of equivalent disturbances/perturbations, since usually in the real tasks it is not previously known, what degree polynomial with a sufficient accuracy approximates the dependence of the unknown coordinate on the random parameters. Therefore degree q of the approximating polynomial is necessary to choose predominantly empirically, via tests/samples and comparisons of the results with each other.

It must be noted that on the labor expense with the low values of m is more preferable the method of equivalent

disturbances/perturbations, and with the high values of m - Monte Carlo method.

Tasks examined above of calculation and accuracy analysis are the essential component part of approaches and methods of the selection of structure and parameters of nonlinear system. For this purpose, as a rule, it is necessary to conduct numerical research of the dependence of the probabilistic characteristics of the accuracy of system on the values of some of its parameters and characteristics. In this case it is necessary to reveal/detect the insignificant changes in the accuracy, which are especially small, if the values of the parameters of system are close to the optimum ones. Changes in the characteristics of accuracy in this case knowingly are considerably less than absolute errors in the method of calculation of accuracy.

The method of equivalent disturbances/perturbations provides the possibility of this type of research, if with a change in the parameters the systems of the realization of random variables $V_j$ are not changed. This condition, obviously, always can be satisfied.

The method for statistical testing can afford this possibility only in such a case, when the realizations of initial random interactions are not changed with a change in the parameters of

system.

Thus, during the use of the Monte Carlo method research of the dependence of the accuracy of system on its parameters should be conducted during one and the same fixed/recorded realizations of the initial disturbances/perturbations, which operate on the system. This draws together the Monte Carlo method with the method of equivalent disturbances/perturbations; the difference between both methods in this case to a considerable extent is erased.

Table XI.1.

| Относительная погрешность в % (1) | N | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 10 | 20 | 30 | 40 | 60 | 80 | 100 | 150 | 200 |
| $\frac{\sigma_{m_y^*}}{\sigma_y^*} \cdot 100$ | 31,6 | 22,4 | 18,2 | 15,8 | 12,9 | 11,2 | 10 | 8,2 | 7,1 |
| $\frac{\sigma_{D_y^*}}{D_y^*} \cdot 100$ | 47,2 | 32,4 | 26,3 | 22,6 | 18,4 | 15,9 | 14,2 | 11,6 | 10,0 |

Key: (1). Relative error %.

Page 375.

Let us pass to the examination of the methods of obtaining on TsVM of the realizations of the random interactions, necessary during the research of the accuracy of nonlinear systems by the Monte Carlo method.

5. Generation of random interactions during the digital simulation SAU.

There are two in principle different approaches to obtaining of random interactions during the simulation SAU and TsVM.

The first approach is based on the use of physical random

phenomena, as a result of which are developed the realizations of the random interactions, which cannot be either previously forecast, or repeated.

With the second approach the program (algorithmic) methods of obtaining the realizations of random interactions are used. It is obvious that these realizations will no longer be, strictly speaking, random: it is possible to previously predict them and to repeat when this is necessary. But these realizations will be equivalent to random ones in one (in this case most important) sense - to them it is possible to use the same methods of statistical processing, that also to the realization of purely random interactions, and with respect to the results of this processing they will be equivalent. Therefore developed in this way realizations are called pseudorandom. Pseudorandom in this case are called interactions themselves in the model.

Usually the simulation of random or pseudorandom variables and functions is made into two stages: are first developed the realizations of numbers, which have the uniform distribution law in the specific fixed/recorded segment, then from them by mathematical transformations are obtained the realizations of random variables and systems of the random variables, which possess the required statistical characteristics.

This order is caused by the fact that has the capability by relatively simple paths to obtain the realizations of random and pseudorandom numbers, which very accurately obey the law uniform probability density in the preset fixed/recorded segment.

The production/consumption/generation (generation) of purely random numbers is realized with the help of the special attachments to the machines, usually called random-number generators [15, 95, 116]. The operating principles of such sensors are based on the use of physical random phenomena, for example, such, as the phenomena of radioactive decay or thermal noises in the electron tubes. At the output of sensors is obtained the binary code, in each bit of which independent of other bits can appear zero or one with the probability, equal to $^1/_2$.

Page 376.

This binary code can be treated as positive binary fixed-point number. It is easy to see that such numbers with a sufficiently large quantity of bits can be considered the realizations of the continuous random variable, subordinated to the uniform distribution law in the range from zero to one.

The fact that for the generation of random number there is required the small time, usually which does not exceed the time of access to OZU of TsVM, is the advantage of the described method.

The generation of pseudorandom numbers, as a rule, is realized by program methods. Many such algorithms and programs [15, 23, 123] are at present known.

The simplest algorithms, easily realized with the help of TsVM, are constructed thus. The at random binary code, which occupies all bits of certain cell OZU TsVM or specific part of its bits, is chosen. Above this code a number of arithmetic and other operations, realized with the help of certain simple machine program, which consists usually of several instructions, is made.

As a result of fulfilling this program is obtained the new value of the code, to which again is applied the same program, etc. Thus is obtained the sequence of the codes, which under specific conditions can be used as the pseudorandom numbers.

The sequence of the $r$-bit codes, beginning from certain number, is always periodic with the period of not more than $2^r$, since it is

completely obvious that not more than through $2^r$ members must again appear one of the years available already, after which in a row all remaining codes in the same order again will begin to be repeated.

So that obtained thus codes it would be possible to use as the pseudorandom numbers, it is necessary to satisfy the following conditions:

the period of sequence must be sufficient to large ones; ¶ terms must not appear repetition of one and the same number;

¶ the statistical law of distribution of the obtained numbers must be with a sufficient accuracy uniform to;

the statistical connections/communications between the members of sequence must be negligible.

Therefore during the use of new algorithms, and sometimes also the new initial codes the obtained sequence of the codes is subjected to research with the help of the special test programs for the satisfaction to the enumerated requirements.

Are worked out the economical programs of the generation of random numbers, which consist in all of ter-penta three-address

instructions, the developing sequences of numbers with period $10^5$-$10^6$.

Page 377.

Let us give some of such methods.

The method of deductions, for the first time used by Lemer [125], is reduced to the formation/education of sequence $\{\alpha_n\}$ on the following recursion relation:

$$\alpha_{n+1} \quad \kappa\alpha_n \pmod{M}, \qquad \text{(XI.25)}$$

i.e., $\alpha_n$ is multiplied to certain number $\kappa$ and then remainder/residue from the division of this product into Mach number is taken.

Are applied and somewhat more complicated algorithms of the type

$$\alpha_{n+1} = a\beta_n;\ \beta_{n+1} = \kappa\beta_n \pmod{M}, \qquad \text{(XI.26)}$$

which give the best results.

The algorithm of the form

$$\alpha_{n+1} = [(2^\kappa + 1)\alpha_n + C] \pmod{2^m}, \qquad \text{(XI.27)}$$

where $\kappa \geq 2$, and C is odd, proposed by Rotenberg [130].

Somewhat more complicated methods give an increase in the period to the virtually infinite values.

A considerable number of algorithms and a vast bibliography on this question are given in the work of D. I. Golenko [23].

In recent years begin to be developed/processed the equipment sensors of pseudorandom numbers, using so-called linear switching circuits [41, 71, 103]. Such sensors are not structurally very complicated and develop the sequences of the pseudorandom R-bit evenly distributed numbers with period $2^r-1$.

Diagrams have high speed operation, and the time of access to them does not exceed the access time to OZU of machine.

As it was said above, from the sequence of the independent evenly distributed random or pseudorandom variables other random or pseudorandom variables and their systems, subordinated to the arbitrary distribution laws, can be manufactured.

Analog quantities $\xi_i$, subordinated to arbitrary differential law allocations p(x), can be obtained, for example, by Neumann's method

[23, 127].

Is extensively used also the method of functional transformation, or the method of inverse functions [16]. A considerable number of methods, which do not carry general character, is worked out for the single frequently encountered distribution laws, such as, for example normal, Rayleigh and so forth [15, 71, 123].

For the systems of random variables, subordinated to the arbitrary distribution laws, it is possible to use the method of Neumann or the method of functional transformation, for example.

Page 378.

During the simulation of nonlinear automatic control systems it is very important to obtain the realizations of random functions. This can be done, as it was mentioned higher, on the base of obtaining random numbers. For this purpose it is possible, for example, to use a canonical expansion of random function, being limited in this case to the final segment of canonical series. The realizations of random function then can be obtained via the substitution of the realizations of random coefficients in the canonical series. For the simulation of the consecutive values of

stationary random processes it is possible to use, for example, following method [90]: is developed the sequence of independent variables $\xi_{\kappa}$, from which they are obtained the value of stationary process with the help of the sliding addition:

$$y_n = \sum_{\kappa=0}^{N} C_\kappa \xi_{n-\kappa}.$$

Are known other methods of obtaining of such processes [51, 61, 81]. Different classes of random processes can be obtained by the transformations of stationary random processes. And finally it is possible to use recordings of real random processes. The use/application of this method is facilitated by the fact that are at present developed the methods of the digital recording of the results of experiment with the use as the carrier of information of magnetic tape, and sometimes punched tape. Such methods of recording are used for processing of the results of experiment on TsVM, so that during the creation of the systems of digital recording is always solved a question of input of the recorded information into the machine.

6. Possibilities of applying TsVM for the search for optimal solutions and selection of the optimum values of the parameters of nonlinear systems.

In the previous paragraphs the methods of the solution of some problems of calculation and analysis of nonlinear systems with the use of a method of simulation on the TsVM were examined. It is now necessary to switch over to the tasks of synthesis, i.e., findings of the optimum versions of systems.

The class of the tasks of synthesis is even more wide, than the class of analysis.

Entire variety of the tasks of synthesis can be divided into two sub-classes.

First, this those tasks, which admit solution by the special receptions/procedures, when from the initial differential equations and the variables it is possible to switch over to others, that allow/assume in this or another sense more effective solution.

Page 379.

In the second place, there are such tasks, where effective analytical procedures indicated above it is impossible to find. Then it is necessary to construct the solution of stated problem with the

use of methods of the simulation of systems, in this case of digital simulation.

The sufficiently wide arsenal of analytical procedures for finding the optimal solutions is at present accumulated. These are variational methods, dynamic programming, the use/application of principle of maximum [74], the theory of optimum dynamic systems [77], different methods of optimum automatic systems [100], etc.

All these methods require the large volume of computational works, if system is not sufficiently simple. For the complex systems usually is required such volume of computational works, which is impracticable even for contemporary TsVM. In present chapter let us pause only at the second sub-class of the tasks, whose solution is based on the use of methods of the simulation of systems.

As it was said above, the task of synthesis in this case also is based on the multiple repetition of the task of calculation with the variation of the conditions of task. According to the results of calculation quality coefficient of system is determined. The variation of the conditions of task must be organized so as, gradually passing from one version to the next, to find version with the best quality.

Quality coefficient must show, how good system satisfies the requirements presented to it. For the numerical definition of quality coefficient it is necessary in the mathematical form to record all those requirements to the system, which must be taken into consideration during the solution of the problem of synthesis. Further on their basis it is necessary to form quality coefficient of system. This formulation of the problem corresponds to the methodology of the theory of operations, and actually the task of the selection of the best version in this setting is a task of the theory of operations.

Let us consider the task of the selection of the optimum version of system in more detail.

To the mathematical formulation of the requirements, presented to the system, only path conducts. It is necessary that each requirement would be imposed on certain numerical characteristics of the system, for which must exist the method of calculating its values under given conditions of task.

Page 380.

Thus, must be determined the method of computing the functional

$$\Phi_i = \Phi_i(x_1, x_2, \ldots, x_n; \psi_1, \psi_2, \ldots, \psi_m),$$

where $\Phi_i$ — numerical system characteristics;

$x_j$ — parameters of the systems, whose values can be varied and must be selected as a result of the solution of problem;

$\psi_k$ — some functions from other independent variables, on form of which also depends the value of characteristic $P_i$; the form of these functions is subject to selection.

There are following most commonly used methods of the formulation of requirements for the numerical values of characteristics $\Phi_i$.

1. Assignment to region of allowed values. This region in the technical tasks usually is not very complicated and can be preset by one or several inequalities, for example:

$\Phi_i > b$ (limitation from below);

$\Phi_i < a$ (limitation on top);

$b<\Phi_i<a$ (limitation from above and from below).

where a>b.

Inequalities can be both the strict and lax.

In the particular case the region of allowed values $\Phi_i$ braces itself into the point, then has the equation

$$\Phi_i=\Phi_i(x_1, x_2, \ldots, x_n;\ \psi_1, \psi_2, \ldots, \psi_m)=a.$$

This equation leads to exception/elimination of one of the parameters $x_i$, and therefore this case we will not examine.

2. Values of characteristic must be greatest (or smallest) from possible ones. This means that the functional

$$\Phi_i=\Phi_i(x_1, x_2, \ldots, x_n;\ \psi_1, \psi_2, \ldots, \psi_m)$$

must achieve in the permissible region the greatest (or smallest) value.

Apparently, all forms of requirements can be reduced to the requirements of the 1st or 2nd kind.

It is most simple and well studied the case, when the requirement of the 2nd kind only one, for example, the requirement of the minimum of the mean square of error. In this case precisely it determines the fundamental idea of the optimization of system. Remaining requirements are further, and their specific logical combination determines region D of the allowed values of the parameters. Then mathematically the task of the search for optimum version is formulated as follows: to determine the greatest (or smallest) value of value $U = F(x_1, x_2, \ldots, x_n; \psi_1, \psi_2, \ldots, \psi_m)$ under the condition of fulfilling certain logical combination of the following inequalities:

$$y_\kappa = f_\kappa(x_1, x_2, \ldots, x_n; \psi_1, \psi_2, \ldots, \psi_m) > 0,$$

where F and $f_\kappa$— functionals from parameters $x_i$ and functions $\psi_j$

As it was indicated above, there are many varieties of the tasks of optimization. Earlier all these varieties were considered isolated/insulated as in the theoretical sense, so also from the point of view of the development of computational algorithms.

Page 381.

The general mathematical theory of such extreme tasks, based on the

concepts of functional analysis, at present is created.

In particular, A. Ya. Dubovitskiy and A. A. Milyutin [28], and also Ye. S. Levitin and B. T. Polyak [59] presented the approach, which makes it possible to uniform obtain the necessary conditions of extremum for the tasks of this type and to investigate questions of existence and uniqueness of extremum. These works have fundamental scientific value in the general theory of extreme tasks.

The formulated above task can be reduced to the task of finding the extremum in the Euclidean space. For this of function $\psi_*$ they are led to the set of certain quantity of parameters, for example, via the approximation of function $\psi_*$ by polynomial. In this case by the parameters, which determine the form of the function $\psi_*$, they can be coefficients of the polynomial. Then the funtions F and $f_*$ will be reduced to the functions of many independent variables, and the task of optimization - to the task of finding the optimum values of the parameters of system. Task will be formulated as follows:

To determine vector X in the n-dimensional space, which communicates the greatest (smallest) value scalar function

$$U = F(X)$$

with fulfilling of the combinations of the conditions

$$y_\kappa = f_\kappa(X) > 0, \ \kappa = 1, 2, \ldots, N. \qquad \text{(XI.28)}$$

where $f_\kappa$ — scalar functions of vector X.

We will for the certainty consider that the optimum version is characterized by the smallest value of function U(X).

This task relates to the tasks of nonlinear programming, and it it is possible to solve by the different methods, which differ in the complexity and the accuracy. For the case in question one of appropriate ones is the method whose idea is stated below. This method, being simple, provides the solution of the problem of finding the optimum values of the parameters of system with the acceptable accuracy and relatively small difficulty of calculations.

Task is reduced to finding of the minimum of certain function of the V(X) point X in the n-dimensional space, the domain of definition of which is unconfined. Let us determine function V(X) so that in that region D of the space, where the necessary combination.of inequalities (XI.28) is made, it would be approximately equal to function U(X), and out of this region differed from U(X) to the side of worsening/deterioration in the quality, moreover this difference must be greater, the more strongly are broken inequalities (XI.28), i.e., the further it goes away the point X in the n-dimensional space

from the boundaries of the region D.

Page 382.

Let us assign function V in the following manner:

$$V(X)=U(X)+Z(X).$$

Function Z(X) is called penalty function. Let us assume that it satisfies the following requirements:

Z(X)>0;

Z(X) is sufficiently small within D region;

Z(X) sharply grows out of D region with the removal/distance of point X from its borders.

D region in general form can be described thus.

For each of conditions (XI.28) let us determine in the space Boolean function $z_x$:

$$z_\kappa = 1, \text{ если } y_\kappa \geqslant 0;$$
$$z_\kappa = 0, \text{ если } y_\kappa < 0.$$

Key: (1). if.

With the help of the operations of Boolean algebra let us determine the Boolean function z from the variables $z_\kappa$, describing the logical combination of inequalities (XI.28):

$$z = f(z_1, z_2, \ldots, z_N) = \Psi(X).$$

This function determines the D region in the space as follows:

$$X \in D, \text{ если } z = \Psi(X) = 1; \quad X \notin D, \text{ если } z = \Psi(X) = 0.$$

Key: (1). if.

In particular, if D region is determined by simultaneous satisfaction of conditions (XI.28), then $z = f(z_1, z_2, \ldots, z_N) \quad z_1 \wedge z_2 \wedge \ldots z_N$.

Let us assume that function f can be expressed with the help of the operations of logical multiplication, logical addition and negation, for example, in the normal disjunctive form. For this case let us give the simple procedure of the composition of penalty function.

Let us introduce into the examination the continuous function Ш(y) of the scalar variable y, which satisfies three conditions

$$Ш(y) > 0;$$
$$Ш(y_1) < ce^{ay};\ a > 0;\ c > 0\ \text{при}^{(1)}\ y < y_1 < 0;$$
$$Ш(y_2) > ce^{by};\ b > 0;\ c > 0\ \text{при}^{(1)}\ y > y_2 > 0.$$

Key: (1) when.

We will call the class of the functions, which satisfy these conditions, the Ш-class.

It is obvious that this function always can be found, whatever a, b, c, $y_1$, $y_2$. It is possible to attain by the corresponding selection its parameters, in order to

$$Z(X) = Ш(y),$$

where $y=f_1(X)$ satisfies the requirements, before(been to the penalty function Z with one condition (XI.28), i.e., with N=1.

Page 383.

Let us designate by symbol ⊙ the following operation above the functions Ш(y):

$$Ш_1(y_1) \odot Ш_2(y_2) = [Ш_1(y_1)^{-1} + Ш_2(y_2)^{-1}]^{-1}.$$

It is possible to easily generalize operation $\odot$ to several functions $Ш_i(y_i)$:

$$Ш_1(y_1) \odot Ш_2(y_2) \odot \ldots \odot Ш_S(y_S) \quad \left[\sum_{i=1}^{S} Ш_i(y_i)^{-1}\right]^{-1};$$

obvious that operation $\odot$ is commutative and associative.

Let us designate by symbol $\overline{Ш(y)}$ function $Ш(y)^{-1}$ or $Ш(-y)$. Let us assume now that scalar the variables y will be the left sides of inequalities (XI.28).

It is possible to claim the following:

1. If

$$z = f(z_1, z_2) = z_1 \vee z_2,$$

then function W(X) of form

$$W(X) = Ш_1(y_1) \odot Ш_2(y_2)$$

with the appropriate selection of the parameters of functions $Ш_1(y_1)$

and $\text{Ш}_2(y_2)$ will satisfy the requirements, presented to the penalty function Z(X) with N=2.

2. If

then function

$$z = f(z_1, z_2) = z_1 \wedge z_2,$$
$$W(X) = \text{Ш}_1(y_1) + \text{Ш}_2(y_2)$$

with appropriate selection of parameters of functions $\text{Ш}_1(y_1)$ and $\text{Ш}_2(y_2)$ will also satisfy the requirements, presented to penalty function Z(X) with N=2.

3. If

then function

$$z = f(z_1) = \bar{z}_1,$$
$$W(X) = \overline{\text{Ш}_1(y_1)}$$

during appropriate selection of its parameters satisfies the requirements, presented to penalty function Z(X) with N=1.

Hence it follows that many specific above logical variable/alternating $z_i$ and all possible Boolean functions from $z_i$,

those formed with the help of the operations of logical addition, logical multiplication and negation, are isomorphic to many functions of the Ш-class from the left sides of inequalities (XI.28) and all possible functions from them, formed with the help of operations $\odot$, $+$ and $^{-}$. formed thus functions W(X) during the appropriate selection of the parameters of functions $Ш_i(y_i)$ satisfying the requirements, presented to the penalty functions Z(X). This gives the possibility to realize the procedure of the construction of penalty functions.

Page 384.

On the basis of the preset logical combination of conditions (XI.28) are chosen Boolean variables $z_i$ and Boolean function $z = f(z_1, z_2, \ldots, z_N)$. From the formula for z we will obtain expression for the penalty function Z( X) path of replacing the variables $z_i$ on function $Ш_i(y_i)$, $z$ on Z(X), symbols $\vee$ on $\odot$ and symbols $\wedge$ on + (symbols $^{-}$ they are retained).

In the obtained equation we reveal the sense of symbols $\odot$ and $^{-}$ and find the unknown expression for the penalty function. After this there remains only to make more precise the values of the parameters of functions $Ш_i(y_i)$ so that the values of function Z(X) would correspond to the requirements presented to it.

For the determination of the minimum of function V(X) it is possible to use a method of gradient [37, 72], the method of steepest descent [36, 86], Box-Wilson method [65], the methods of random search [82], simplex method [115], the method of ravines [21], etc. All these methods are realized in the digital computers.

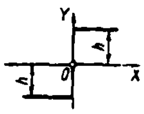
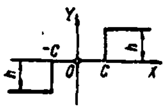
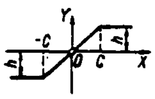
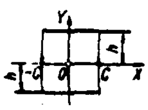
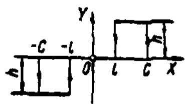
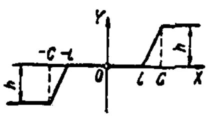
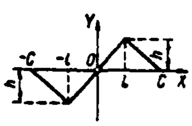
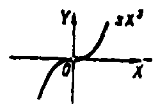
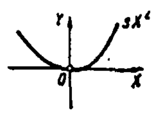
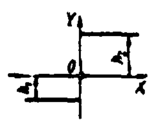
Experiment confirms the possibility of the automation of the selection of optimal solutions for any systems, if this does not require the excessive volumes of computational work.

The automated method of optimization makes it possible to lay the analysis of the results of calculations, the selection of the new values of the varied parameters, their gradual improvement and fulfillment according to them of the check calculations on the machine. Thus, is improved the quality of the development of automatic control systems and the expenditures of time for their designing considerably are reduced.

Page 385.

Appendices.

Page 386. APPENDIX 1. Nonlinear characteristics.

| (1) № по пор. | (2) Нелинейность | $Y = F(X)$ |
|---|---|---|
| 1 |  | $Y = h$; $X > 0$; $Y = -h$; $X < 0$ |
| 2 |  | $Y = h$; $X > C$; $Y = 0$; $\lvert X \rvert \leqslant C$; $Y = -h$; $X < -C$ |
| 3 |  | $Y = h$; $X > C$; $Y = \frac{h}{C} X$; $\lvert X \rvert \leqslant C$; $Y = -h$; $X < -C$ |
| 4 |  | $Y = h$; $X > C$; $\dot{X} \leqslant 0$; $Y = h$; $-C < X < C$; $\dot{X} < 0$; $Y = -h$; $-C < X < C$; $\dot{X} > 0$; $Y = -h$; $X < -C$; $\dot{X} \geqslant 0$ |
| 5 |  | $Y = h$; $X > C$; $\dot{X} \geqq 0$; $Y = h$; $l < X < C$; $\dot{X} < 0$; $Y = 0$; $l < X < C$; $\dot{X} > 0$; $Y = 0$; $\lvert X \rvert < l$; $\dot{X} \geqq 0$; $Y = 0$; $-C < X < -l$; $\dot{X} < 0$; $Y = -h$; $-C < X < -l$; $\dot{X} > 0$; $Y = -h$; $X < -C$; $\dot{X} \leqslant 0$ |
| 6 |  | $Y = h$; $X > C$; $Y = \frac{h}{C-l} \times (X - l)$; $l < X < C$; $Y = 0$; $\lvert X \rvert \leqslant l$; $Y = \frac{h}{C-l} \times (X + l)$; $-C < X < -l$; $Y = -h$; $X < -C$ |
| 7 |  | $Y = h - \frac{h}{C-h} \times (X - l)$; $l < X < C$; $Y = \frac{h}{l} X$; $\lvert X \rvert \leqslant l$; $Y = -h + \frac{h}{C-h} \times (X + l)$; $-C < X < -l$; $Y = 0$; $\lvert X \rvert \geqslant C$ |
| 8 |  | $Y = sX^3$ |
| 9 |  | $Y = sX^3$ |
| 10 |  | $Y = +h_2$; $X > 0$; $Y = -h_1$; $X < 0$ |

Page 387.

Appendix 2. Formulas for the statistical characteristics and the factors of amplification of nonlinearity.

Designations:

1. The statistical linearization

$F_0$ - the statistical characteristic of nonlinearity;

$k_0$ - statistical amplification factor on the mathematical expectation;

$k_1$ - statistical factor of amplification of fluctuations;

$k_2$ - statistical factor of amplification of derived fluctuations;

2. The combined statistical and harmonic linearization

$F_0^*$ - the average/mean statistical characteristic of nonlinearity;

a* - first statistical harmonic factor of amplification of

nonlinearity;

b* - second statistical harmonic factor of amplification of nonlinearity;

$k_0^*$ - average/mean statistical amplification factor of mathematical expectation;

$k_1^*$ - average statistical factor of amplification of fluctuations;

$k_2^*$ - average statistical factor of amplification of derived fluctuations.

Pages 388-392.

(1) *Нелинейность* 1

$$F_0 = k_0 m_X = 2h\Phi\left(\frac{m_X}{\sqrt{D_X}}\right);$$

$$k_1 = \frac{2h}{\sqrt{2\pi D_X}} e^{-\frac{m_X^2}{2D_X}};$$

$$a^* = \frac{h}{x_m} B_0(\alpha);\ b^* = 0 \text{ при } m_X + x_0 = 0;$$

$$k_1^* = \frac{h}{\sqrt{D_X}} C_0(\alpha);\ k_2^* = 0 \text{ при } m_X + x_0 = 0,$$

(3) где

$$\alpha = \frac{x_m}{\sqrt{2D_X}},$$

(4) а функции

$$B_n(\alpha) = \frac{4}{\sqrt{\pi}} \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa+n}[2(\kappa+n)]!}{(2n)!(\kappa!)^2(\kappa+n)!} \left(\frac{\alpha}{2}\right)^{2\kappa+1};$$

$$C_n(\alpha) = \sqrt{\frac{2}{\pi}} \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa+n}[2(\kappa+n)]!}{(2n)!(\kappa!)^2(\kappa+n)!} \left(\frac{\alpha}{2}\right)^{2\kappa};$$

$$n = 0, 1, \ldots$$

(1) *Нелинейность* 2

$$F_0 = k_0 m_X = h\left[\Phi\left(\frac{1+m_1}{\sigma_1}\right) - \Phi\left(\frac{1-m_1}{\sigma_1}\right)\right];$$

$$k_1 = \frac{h}{\sqrt{2\pi D_X}}\left[e^{-\frac{1}{2}\left(\frac{1+m_1}{\sigma_1}\right)^2} + e^{-\frac{1}{2}\left(\frac{1-m_1}{\sigma_1}\right)^2}\right];$$

$$a^* = \frac{h}{x_m} \sum_{n=0}^{\infty} x_1^{2n} B_n(\alpha);$$

$$b^* = 0 \text{ при } m_X + x_0 = 0;$$

$$k_1^* = \frac{h}{\sqrt{D_X}} \sum_{n=0}^{\infty} x_1^{2n} C_n(\alpha);$$

$$k_2^* = 0 \text{ при } m_X + x_0 = 0.$$

где

$$m_1 = \frac{m_X}{C};\ \sigma_1 = \frac{\sqrt{D_X}}{C};$$

$$x_1 = \frac{x_m}{\sqrt{2D_X}}.$$

*Нелинейность 3*

$$F_0 = k_0 m_X = h\left\{(1+m_1)\Phi\left(\frac{1+m_1}{\sigma_1}\right) - (1-m_1)\Phi\left(\frac{1-m_1}{\sigma_1}\right) + \right.$$

$$\left. + \frac{\sigma_1}{\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left(\frac{1+m_1}{\sigma_1}\right)^2} - e^{-\frac{1}{2}\left(\frac{1-m_1}{\sigma_1}\right)^2}\right]\right\};$$

$$k_1 = \frac{h}{C}\left[\Phi\left(\frac{1+m_1}{\sigma_1}\right) + \Phi\left(\frac{1-m_1}{\sigma_1}\right)\right];$$

$$a^* = \frac{h}{C}\sum_{n=0}^{\infty}\frac{x_1^{2n}}{2n+1}B_n(\alpha);\quad b^* = 0 \text{ при } m_X + x_0 = 0;$$

$$k_1^* = \frac{h}{\sqrt{D_X}}\sum_{n=0}^{\infty}\frac{x_1^{2n}}{2n+1}C_n(\alpha);\quad k_2^* = 0 \text{ при } m_X + x_0 = 0.$$

*Нелинейность 4*

$$F_0 = h\left\{\Phi\left(\frac{C+m_X}{\sqrt{D_{XX}}}\right) - \Phi\left(\frac{C-m_X}{\sqrt{D_{XX}}}\right) - 2\Phi\left(\frac{m_{\dot X}}{\sqrt{D_{\dot X\dot X}}}\right)\left[\Phi\left(\frac{C-m_X}{\sqrt{D_{XX}}}\right) + \right.\right.$$

$$\left.\left. + \Phi\left(\frac{C+m_X}{\sqrt{D_{XX}}}\right)\right] + \frac{D_{X\dot X}}{\pi\sqrt{D_{XX}D_{\dot X\dot X}}}\left[e^{-\frac{(C-m_X)^2}{2D_{XX}}} - e^{-\frac{(C+m_X)^2}{2D_{XX}}}\right]\right\};$$

$$k_1 = h\left\{\left[\frac{1}{\sqrt{2\pi D_{XX}}}\left(1 + 2\Phi\left(\frac{m_{\dot X}}{\sqrt{D_{\dot X\dot X}}}\right)\right) + \frac{D_{X\dot X}(C-m_X)}{D_{XX}\pi\sqrt{D_{XX}D_{\dot X\dot X}}}\right]\times\right.$$

$$\times e^{-\frac{(C-m_X)^2}{2D_{XX}}} + \left[\frac{1}{\sqrt{2\pi D_{XX}}}\left(1 - 2\Phi\left(\frac{m_{\dot X}}{\sqrt{D_{\dot X\dot X}}}\right)\right) + \right.$$

$$\left. + \frac{D_{X\dot X}(C+m_X)}{D_{XX}\pi\sqrt{D_{XX}D_{\dot X\dot X}}}\right]e^{-\frac{(C+m_X)^2}{2D_{XX}}};$$

$$k_2 = -\frac{2h}{\sqrt{2\pi D_{\dot X\dot X}}}\, e^{-\frac{m_{\dot X}^2}{2D_{\dot X\dot X}}}\left[\Phi\left(\frac{C-m_X}{\sqrt{D_{XX}}}\right)+\Phi\left(\frac{C+m_X}{\sqrt{D_{XX}}}\right)\right].$$

(1)

*Нелинейность 5*

$$F_0 = k_0 m_X = h\left\{\left[\Phi\left(\frac{C+m_X}{\sqrt{D_{XX}}}\right)-\Phi\left(\frac{l-m_X}{\sqrt{D_{XX}}}\right)\right]\left[\frac{1}{2}-\Phi\left(\frac{m_{\dot X}}{\sqrt{D_{\dot X\dot X}}}\right)\right]+\right.$$

$$+\left[\Phi\left(\frac{l+m_X}{\sqrt{D_{XX}}}\right)-\Phi\left(\frac{C-m_X}{\sqrt{D_{XX}}}\right)\right]\times\left[\frac{1}{2}+\Phi\left(\frac{m_{\dot X}}{\sqrt{D_{\dot X\dot X}}}\right)\right]+$$

$$+\frac{D_{X\dot X}}{2\pi\sqrt{D_{XX}D_{\dot X\dot X}}}\, e^{-\frac{m_{\dot X}^2}{2D_{\dot X\dot X}}}\left[e^{-\frac{(C+m_X)^2}{2D_{XX}}}-e^{-\frac{(l+m_X)^2}{2D_{XX}}}+e^{-\frac{(C-m_X)^2}{2D_{XX}}}-\right.$$

$$\left.\left.-e^{-\frac{(l-m_X)^2}{2D_{XX}}}\right]\right\};$$

$$k_X = k_1 = \frac{h}{\sqrt{2\pi D_{XX}}}\left\{\left[e^{-\frac{(C+m_X)^2}{2D_{XX}}}-e^{-\frac{(l-m_X)^2}{2D_{XX}}}\right]\left[\frac{1}{2}-\Phi\left(\frac{m_{\dot X}}{\sqrt{D_{\dot X\dot X}}}\right)\right]+\right.$$

$$\left.+\left[e^{-\frac{(l+m_X)^2}{2D_{XX}}}-e^{-\frac{(C-m_X)^2}{2D_{XX}}}\right]\left[\frac{1}{2}+\Phi\left(\frac{m_{\dot X}}{\sqrt{D_{\dot X\dot X}}}\right)\right]\right\}+$$

$$+\frac{hD_{X\dot X}}{2\pi\sqrt{D_{XX}D_{\dot X\dot X}}}\, e^{-\frac{m_{\dot X}^2}{2D_{\dot X\dot X}}}\left[\frac{l+m_X}{D_{XX}}\, e^{-\frac{(l+m_X)^2}{2D_{XX}}}-\frac{C+m_X}{D_{XX}}\, e^{-\frac{(C+m_X)^2}{2D_{XX}}}+\right.$$

$$\left.+\frac{C-m_X}{D_{XX}}\, e^{-\frac{(C-m_X)^2}{2D_{XX}}}-\frac{l-m_X}{D_{XX}}\, e^{-\frac{(l-m_X)^2}{2D_{XX}}}\right];$$

$$k_{\dot X} = k_2 = \frac{h}{\sqrt{2\pi D_{\dot X\dot X}}}\, e^{-\frac{m_{\dot X}^2}{2D_{\dot X\dot X}}}\left[\Phi\left(\frac{C+m_X}{\sqrt{D_{XX}}}\right)-\Phi\left(\frac{l-m_X}{\sqrt{D_{XX}}}\right)+\right.$$

$$\left.+\Phi\left(\frac{l+m_X}{\sqrt{D_{XX}}}\right)-\Phi\left(\frac{C-m_X}{\sqrt{D_{XX}}}\right)\right]+\frac{hD_{X\dot X}}{2\pi\sqrt{D_{XX}D_{\dot X\dot X}}}\,\frac{m_{\dot X}}{D_{\dot X\dot X}}\, e^{-\frac{m_{\dot X}^2}{2D_{\dot X\dot X}}}\times$$

$$\times\left[e^{-\frac{(C+m_X)^2}{2D_{XX}}}-e^{-\frac{(l+m_X)^2}{2D_{XX}}}+e^{-\frac{(C-m_X)^2}{2D_{XX}}}-e^{-\frac{(l-m_X)^2}{2D_{XX}}}\right].$$

1 *Нелинейность 6*

$$F_0 = k_0 m_X = \frac{h}{1-\nu}\left\{(1+m_1)\,\Phi\left(\frac{1+m_1}{\sigma_1}\right) - (1-m_1)\,\Phi\left(\frac{1-m_1}{\sigma_1}\right) - \right.$$

$$- (m_1+\nu)\,\Phi\left(\frac{m_1+\nu}{\sigma_1}\right) + (m_1-\nu)\,\Phi\left(\frac{m_1-\nu}{\sigma_1}\right) + \frac{\sigma_1}{\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left(\frac{1+m_1}{\sigma_1}\right)^2} - \right.$$

$$\left.\left. - e^{-\frac{1}{2}\left(\frac{1-m_1}{\sigma_1}\right)^2} - e^{-\frac{1}{2}\left(\frac{\nu+m_1}{\sigma_1}\right)^2} + e^{-\frac{1}{2}\left(\frac{\nu-m_1}{\sigma_1}\right)^2}\right]\right\};$$

$$k_1 = \frac{h}{C}\cdot\frac{1}{1-\nu}\left[\Phi\left(\frac{1+m_1}{\sigma_1}\right) + \Phi\left(\frac{1-m_1}{\sigma_1}\right) - \Phi\left(\frac{m_1+\nu}{\sigma_1}\right) + \Phi\left(\frac{m_1-\nu}{\sigma_1}\right)\right],$$

где 3

$$\nu = \frac{l}{C};\; m_1 = \frac{m_X}{C};\; \sigma_1 = \frac{\sqrt{D_X}}{C}.$$

1 *Нелинейность 7*

$$F_0 = k_0 m_X = h\left\{\frac{\nu+m_1}{1-\nu}\,\Phi\left(\frac{\nu+m_1}{\sigma_1}\right) - \frac{1+m_1}{1-\nu}\,\Phi\left(\frac{1+m_1}{\sigma_1}\right) - \right.$$

$$- \frac{\nu-m_1}{1-\nu}\,\Phi\left(\frac{\nu-m_1}{\sigma_1}\right) + \frac{1-m_1}{1-\nu}\,\Phi\left(\frac{1-m_1}{\sigma_1}\right) + (\nu+m_1)\times$$

$$\times\Phi\left(\frac{\nu+m_1}{\sigma_1}\right) - (\nu-m_1)\,\Phi\left(\frac{\nu-m_1}{\sigma_1}\right) + \frac{\sigma_1}{\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left(\frac{\nu+m_1}{\sigma_1}\right)^2} - \right.$$

$$\left. - e^{-\frac{1}{2}\left(\frac{\nu-m_1}{\sigma_1}\right)^2}\right] + \frac{\sigma_1}{\sqrt{2\pi}\,(1-\nu)}\left[e^{-\frac{1}{2}\left(\frac{\nu+m_1}{\sigma_1}\right)^2} - e^{-\frac{1}{2}\left(\frac{\nu-m_1}{\sigma_1}\right)^2} - \right.$$

$$\left.\left. - e^{-\frac{1}{2}\left(\frac{1+m_1}{\sigma_1}\right)^2} + e^{-\frac{1}{2}\left(\frac{1-m_1}{\sigma_1}\right)^2}\right]\right\};$$

$$k_1 = \frac{h}{\sqrt{D_X}}\left\{\left[\Phi\left(\frac{\nu+m_1}{\sigma_1}\right) + \Phi\left(\frac{\nu-m_1}{\sigma_1}\right)\right]\frac{m_1^2+\sigma_1^2}{\sigma_1^2} + \right.$$

$$+ \left[\Phi\left(\frac{\nu+m_1}{\sigma_1}\right) - \Phi\left(\frac{1+m_1}{\sigma_1}\right)\right]\frac{m_1^2+\sigma_1^2+m_1}{(1-\nu)\,\sigma_1^2} + \left[\Phi\left(\frac{\nu-m_1}{\sigma_1}\right) - \right.$$

$$\left. - \Phi\left(\frac{1-m_1}{\sigma_1}\right)\right]\frac{m_1^2+\sigma_1^2-m_1}{(1-\nu)\,\sigma_1^2} + \frac{1}{(1-\nu)\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left(\frac{\nu+m_1}{\sigma_1}\right)^2} - \right.$$

$$\left. - e^{-\frac{1}{2}\left(\frac{\nu-m_1}{\sigma_1}\right)^2} - e^{-\frac{1}{2}\left(\frac{1+m_1}{\sigma_1}\right)^2} + e^{-\frac{1}{2}\left(\frac{1-m_1}{\sigma_1}\right)^2}\right] +$$

$$\left. + \frac{m_1}{\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left(\frac{\nu+m_1}{\sigma_1}\right)^2} - e^{-\frac{1}{2}\left(\frac{\nu-m_1}{\sigma_1}\right)^2}\right] - \frac{F_0 m_1}{h\sigma_1}\right\}.$$

(1) Нелинейность 8

$$F_0 = k_0 m_X = s m_X D_X \left[3 + \frac{m_X^2}{D_X}\right];$$

$$k_1 = 3sD_X \left[1 + \frac{m_X^2}{D_X}\right];$$

$$a^* = \frac{3sx_m^2}{4};\ b^* = 0 \ \text{(2) при}\ m_X + x_0 = 0;$$

$$k_1^* = \frac{3}{2} s\,(x_m^2 + 2D_X);\ k_2^* = 0 \ \text{(2) при}\ m_X + x_0 = 0.$$

(1) Нелинейность 9

$$F_0 = sD_X \left(1 + \frac{m_X^2}{D_X}\right);$$

$$k_1 = 2sm_X.$$

(1) Нелинейность 10

$$F_0 = \frac{1}{2}(h_2 - h_1) + (h_2 + h_1)\,\Phi\left(\frac{m_X}{\sqrt{D_X}}\right);$$

$$k_1 = \frac{h_2 + h_1}{\sqrt{2\pi D_X}}\, e^{-\frac{m_X^2}{2D_X}}.$$

Key: (1). Nonlinearity.

FOOTNOTE [1]. Nonlinearities are numbered in accordance with application/appendix 1. ENDFOOTNOTE.

(2). with. (3). where. (4). and function.

Page 393.

Appendix 3. Values of function $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{t^2}{2}}dt$

| $x$ | $\Phi(x)$ | $x$ | $\Phi(x)$ | $x$ | $\Phi(x)$ | $x$ | $\Phi(x)$ |
|---|---|---|---|---|---|---|---|
| 0,00 | 0,0000 | 0,26 | 0,1026 | 0,52 | 0,1985 | 0,78 | 0,2823 |
| 0,01 | 0,0040 | 0,27 | 0,1064 | 0,53 | 0,2019 | 0,79 | 0,2852 |
| 0,02 | 0,0080 | 0,28 | 0,1103 | 0,54 | 0,2054 | 0,80 | 0,2881 |
| 0,03 | 0,0120 | 0,29 | 0,1141 | 0,55 | 0,2088 | 0,81 | 0,2910 |
| 0,04 | 0,0160 | 0,30 | 0,1179 | 0,56 | 0,2123 | 0,82 | 0,2939 |
| 0,05 | 0,0199 | 0,31 | 0,1217 | 0,57 | 0,2157 | 0,83 | 0,2967 |
| 0,06 | 0,0239 | 0,32 | 0,1255 | 0,58 | 0,2190 | 0,84 | 0,2995 |
| 0,07 | 0,0279 | 0,33 | 0,1293 | 0,59 | 0,2224 | 0,85 | 0,3023 |
| 0,08 | 0,0319 | 0,34 | 0,1331 | 0,60 | 0,2257 | 0,86 | 0,3051 |
| 0,09 | 0,0359 | 0,35 | 0,1368 | 0,61 | 0,2291 | 0,87 | 0,3078 |
| 0,10 | 0,0398 | 0,36 | 0,1406 | 0,62 | 0,2324 | 0,88 | 0,3106 |
| 0,11 | 0,0438 | 0,37 | 0,1443 | 0,63 | 0,2357 | 0,89 | 0,3133 |
| 0,12 | 0,0478 | 0,38 | 0,1480 | 0,64 | 0,2389 | 0,90 | 0,3159 |
| 0,13 | 0,0517 | 0,39 | 0,1517 | 0,65 | 0,2422 | 0,91 | 0,3186 |
| 0,14 | 0,0557 | 0,40 | 0,1554 | 0,66 | 0,2454 | 0,92 | 0,3212 |
| 0,15 | 0,0596 | 0,41 | 0,1591 | 0,67 | 0,2486 | 0,93 | 0,3238 |
| 0,16 | 0,0636 | 0,42 | 0,1628 | 0,68 | 0,2517 | 0,94 | 0,3264 |
| 0,17 | 0,0675 | 0,43 | 0,1664 | 0,69 | 0,2549 | 0,95 | 0,3289 |
| 0,18 | 0,0714 | 0,44 | 0,1700 | 0,70 | 0,2580 | 0,96 | 0,3315 |
| 0,19 | 0,0753 | 0,45 | 0,1736 | 0,71 | 0,2611 | 0,97 | 0,3340 |
| 0,20 | 0,0793 | 0,46 | 0,1772 | 0,72 | 0,2642 | 0,98 | 0,3365 |
| 0,21 | 0,0832 | 0,47 | 0,1808 | 0,73 | 0,2673 | 0,99 | 0,3389 |
| 0,22 | 0,0871 | 0,48 | 0,1844 | 0,74 | 0,2703 | 1,00 | 0,3413 |
| 0,23 | 0,0910 | 0,49 | 0,1879 | 0,75 | 0,2734 | 1,01 | 0,3438 |
| 0,24 | 0,0948 | 0,50 | 0,1915 | 0,76 | 0,2764 | 1,02 | 0,3461 |
| 0,25 | 0,0987 | 0,51 | 0,1950 | 0,77 | 0,2794 | 1,03 | 0,3485 |

Page 394.

Continuation appendix 3.

| x | Φ(x) | x | Φ(x) | x | Φ(x) | x | Φ(x) |
|---|---|---|---|---|---|---|---|
| 1,04 | 0,3508 | 1,34 | 0,4099 | 1,64 | 0,4495 | 1,94 | 0,4738 |
| 1,05 | 0,3531 | 1,35 | 0,4115 | 1,65 | 0,4505 | 1,95 | 0,4744 |
| 1,06 | 0,3554 | 1,36 | 0,4131 | 1,66 | 0,4515 | 1,96 | 0,4750 |
| 1,07 | 0,3577 | 1,37 | 0,4147 | 1,67 | 0,4525 | 1,97 | 0,4756 |
| 1,08 | 0,3599 | 1,38 | 0,4162 | 1,68 | 0,4535 | 1,98 | 0,4761 |
| 1,09 | 0,3621 | 1,39 | 0,4177 | 1,69 | 0,4545 | 1,99 | 0,4767 |
| 1,10 | 0,3643 | 1,40 | 0,4192 | 1,70 | 0,4554 | 2,00 | 0,4772 |
| 1,11 | 0,3665 | 1,41 | 0,4207 | 1,71 | 0,4564 | 2,02 | 0,4783 |
| 1,12 | 0,3686 | 1,42 | 0,4222 | 1,72 | 0,4573 | 2,04 | 0,4793 |
| 1,13 | 0,3708 | 1,43 | 0,4236 | 1,73 | 0,4582 | 2,06 | 0,4803 |
| 1,14 | 0,3729 | 1,44 | 0,4251 | 1,74 | 0,4591 | 2,08 | 0,4812 |
| 1,15 | 0,3749 | 1,45 | 0,4265 | 1,75 | 0,4599 | 2,10 | 0,4821 |
| 1,16 | 0,3770 | 1,46 | 0,4279 | 1,76 | 0,4608 | 2,12 | 0,4830 |
| 1,17 | 0,3790 | 1,47 | 0,4292 | 1,77 | 0,4616 | 2,14 | 0,4838 |
| 1,18 | 0,3810 | 1,48 | 0,4306 | 1,78 | 0,4625 | 2,16 | 0,4846 |
| 1,19 | 0,3830 | 1,49 | 0,4319 | 1,79 | 0,4633 | 2,18 | 0,4854 |
| 1,20 | 0,3849 | 1,50 | 0,4332 | 1,80 | 0,4641 | 2,20 | 0,4861 |
| 1,21 | 0,3869 | 1,51 | 0,4345 | 1,81 | 0,4649 | 2,22 | 0,4868 |
| 1,22 | 0,3888 | 1,52 | 0,4357 | 1,82 | 0,4656 | 2,24 | 0,4875 |
| 1,23 | 0,3907 | 1,53 | 0,4370 | 1,83 | 0,4664 | 2,26 | 0,4881 |
| 1,24 | 0,3925 | 1,54 | 0,4382 | 1,84 | 0,4671 | 2,28 | 0,4887 |
| 1,25 | 0,3944 | 1,55 | 0,4394 | 1,85 | 0,4678 | 2,30 | 0,4893 |
| 1,26 | 0,3962 | 1,56 | 0,4406 | 1,86 | 0,4686 | 2,32 | 0,4898 |
| 1,27 | 0,3980 | 1,57 | 0,4418 | 1,87 | 0,4693 | 2,34 | 0,4904 |
| 1,28 | 0,3997 | 1,58 | 0,4429 | 1,88 | 0,4699 | 2,36 | 0,4909 |
| 1,29 | 0,4015 | 1,59 | 0,4441 | 1,89 | 0,4706 | 2,38 | 0,4913 |
| 1,30 | 0,4032 | 1,60 | 0,4952 | 1,90 | 0,4713 | 2,40 | 0,4918 |
| 1,31 | 0,4049 | 1,61 | 0,4463 | 1,91 | 0,4719 | 2,42 | 0,4922 |
| 1,32 | 0,4066 | 1,62 | 0,4474 | 1,92 | 0,4726 | 2,44 | 0,4927 |
| 1,33 | 0,4082 | 1,63 | 0,4484 | 1,93 | 0,4732 | 2,46 | 0,4931 |
| 2,48 | 0,4934 | 2,66 | 0,4961 | 2,84 | 0,4977 | 3,00 | 0,49865 |
| 2,50 | 0,4938 | 2,68 | 0,4963 | 2,86 | 0,4979 | 3,20 | 0,49931 |
| 2,52 | 0,4941 | 2,70 | 0,4965 | 2,88 | 0,4980 | 3,40 | 0,49966 |
| 2,54 | 0,4945 | 2,72 | 0,4967 | 2,90 | 0,4981 | 3,60 | 0,499841 |
| 2,56 | 0,4948 | 2,74 | 0,4969 | 2,92 | 0,4982 | 3,80 | 0,499928 |
| 2,58 | 0,4951 | 2,76 | 0,4971 | 2,94 | 0,4984 | 4,00 | 0,499968 |
| 2,60 | 0,4953 | 2,78 | 0,4973 | 2,96 | 0,4985 | 4,50 | 0,499997 |
| 2,62 | 0,4956 | 2,80 | 0,4974 | 2,98 | 0,4986 | 5,00 | 0,49999997 |
| 2,64 | 0,4959 | 2,82 | 0,4976 | | | | |

Page 395.

Appendix 4. Values of function $\Phi'(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0,0 | 0,3989 | 3989 | 3989 | 3988 | 3986 | 3984 | 3982 | 3980 | 3977 | 3973 |
| 0,1 | 3970 | 3965 | 3961 | 3956 | 3951 | 3945 | 3939 | 3932 | 3925 | 3918 |
| 0,2 | 3910 | 3902 | 3894 | 3885 | 3876 | 3867 | 3857 | 3847 | 3836 | 3825 |
| 0,3 | 3814 | 3802 | 3790 | 3778 | 3765 | 3752 | 3739 | 3726 | 3712 | 3697 |
| 0,4 | 3683 | 3668 | 3653 | 3637 | 3621 | 3605 | 3589 | 3572 | 3555 | 3538 |
| 0,5 | 3521 | 3503 | 3485 | 3467 | 3448 | 3429 | 3410 | 3391 | 3372 | 3352 |
| 0,6 | 3332 | 3312 | 3292 | 3271 | 3251 | 3230 | 3209 | 3187 | 3166 | 3144 |
| 0,7 | 3123 | 3101 | 3079 | 3056 | 3034 | 3011 | 2989 | 2966 | 2943 | 2920 |
| 0,8 | 2897 | 2874 | 2850 | 2827 | 2803 | 2780 | 2756 | 2732 | 2709 | 2685 |
| 0,9 | 2661 | 2637 | 2613 | 2589 | 2565 | 2541 | 2516 | 2492 | 2468 | 2444 |
| 1,0 | 2420 | 2396 | 2371 | 2347 | 2323 | 2299 | 2275 | 2251 | 2227 | 2203 |
| 1,1 | 2179 | 2155 | 2131 | 2107 | 2083 | 2059 | 2036 | 2012 | 1989 | 1965 |
| 1,2 | 1942 | 1919 | 1895 | 1872 | 1849 | 1826 | 1804 | 1781 | 1758 | 1736 |
| 1,3 | 1714 | 1691 | 1669 | 1647 | 1626 | 1604 | 1582 | 1561 | 1539 | 1518 |
| 1,4 | 1497 | 1476 | 1456 | 1435 | 1415 | 1394 | 1374 | 1354 | 1334 | 1315 |
| 1,5 | 1295 | 1276 | 1257 | 1238 | 1219 | 1200 | 1182 | 1163 | 1145 | 1127 |
| 1,6 | 1109 | 1092 | 1074 | 1057 | 1040 | 1023 | 1006 | 0989 | 0973 | 0957 |
| 1,7 | 0940 | 0925 | 0909 | 0893 | 0878 | 0863 | 0848 | 0833 | 0818 | 0804 |
| 1,8 | 0790 | 0775 | 0761 | 0748 | 0734 | 0721 | 0707 | 0694 | 0681 | 0669 |
| 1,9 | 0656 | 0644 | 0632 | 0620 | 0608 | 0596 | 0584 | 0573 | 0562 | 0551 |
| 2,0 | 0540 | 0529 | 0519 | 0508 | 0498 | 0488 | 0478 | 0468 | 0459 | 0449 |
| 2,1 | 0440 | 0431 | 0422 | 0413 | 0404 | 0396 | 0387 | 0379 | 0371 | 0363 |
| 2,2 | 0355 | 0347 | 0339 | 0332 | 0325 | 0317 | 0310 | 0303 | 0297 | 0290 |
| 2,3 | 0283 | 0277 | 0270 | 0264 | 0258 | 0252 | 0246 | 0241 | 0235 | 0229 |
| 2,4 | 0224 | 0219 | 0213 | 0208 | 0203 | 0198 | 0194 | 0189 | 0184 | 0180 |
| 2,5 | 0175 | 0171 | 0167 | 0163 | 0158 | 0154 | 0151 | 0147 | 0143 | 0139 |
| 2,6 | 0136 | 0132 | 0129 | 0126 | 0122 | 0119 | 0116 | 0113 | 0110 | 0107 |
| 2,7 | 0104 | 0101 | 0099 | 0096 | 0093 | 0091 | 0088 | 0086 | 0084 | 0081 |
| 2,8 | 0079 | 0077 | 0075 | 0073 | 0071 | 0069 | 0067 | 0065 | 0063 | 0061 |
| 2,9 | 0060 | 0058 | 0056 | 0055 | 0053 | 0051 | 0050 | 0048 | 0047 | 0046 |
| 3,0 | 0044 | 0043 | 0042 | 0040 | 0039 | 0038 | 0037 | 0036 | 0035 | 0034 |
| 3,1 | 0033 | 0032 | 0031 | 0030 | 0029 | 0028 | 0027 | 0026 | 0025 | 0025 |
| 3,2 | 0024 | 0023 | 0022 | 0022 | 0021 | 0020 | 0020 | 0019 | 0018 | 0018 |
| 3,3 | 0017 | 0017 | 0016 | 0016 | 0015 | 0015 | 0014 | 0014 | 0013 | 0013 |
| 3,4 | 0012 | 0012 | 0012 | 0011 | 0011 | 0010 | 0010 | 0010 | 0009 | 0009 |
| 3,5 | 0009 | 0008 | 0008 | 0008 | 0008 | 0007 | 0007 | 0007 | 0007 | 0006 |
| 3,6 | 0006 | 0006 | 0006 | 0005 | 0005 | 0005 | 0005 | 0005 | 0005 | 0004 |
| 3,7 | 0004 | 0004 | 0004 | 0004 | 0004 | 0004 | 0003 | 0003 | 0003 | 0003 |
| 3,8 | 0003 | 0003 | 0003 | 0003 | 0002 | 0002 | 0002 | 0002 | 0002 | 0002 |
| 3,9 | 0002 | 0002 | 0002 | 0002 | 0002 | 0002 | 0002 | 0002 | 0001 | 0001 |

Page 396.

Appendix 5. Formulas for the integrals of the rational-linear functions.

$$I_n = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{g_n(i\omega)}{h_n(i\omega)\,h(-i\omega)}\,d\omega,$$

where

$$h_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n;$$
$$g_n(x) = b_0x^{2n-2} + b_1x^{2n-4} + \cdots + b_{n-1};$$

$$I_1 = \frac{b_0}{2a_0a_1};\quad I_2 = \frac{-b_0 + \dfrac{a_0b_1}{a_2}}{2a_0a_1};$$

$$I_3 = \frac{-a_2b_0 + a_0b_1 - \dfrac{a_0a_1b_2}{a_3}}{2a_0(a_0a_3 - a_1a_2)};$$

$$I_4 = \frac{b_0(-a_1a_4 + a_2a_3) - a_0a_3b_1 + a_0a_1b_2 + \dfrac{a_0b_3}{a_4}(a_0a_3 - a_1a_2)}{2a_0\left(a_0a_3^2 + a_1^2a_4 - a_1a_2a_3\right)};$$

$$I_5 = \frac{M_5}{2a_0\Delta_5};$$

$$M_5 = b_0\left(-a_0a_4a_5 + a_1a_4^2 + a_2^2a_5 - a_2a_3a_4\right) + a_0b_1\left(-a_2a_5 + a_3a_4\right) + a_0b_2\left(a_0a_5 - a_1a_4\right) + a_0b_3\left(-a_0a_3 + a_1a_2\right) + \frac{a_0b_4}{a_5}\times$$
$$\times\left(-a_0a_1a_5 + a_0a_3^2 + a_1^2a_4 - a_1a_2a_3\right);$$

$$\Delta_5 = a_0^2a_5^2 - 2a_0a_1a_4a_5 - a_0a_2a_3a_5 + a_0a_3^2a_4 + a_1^2a_4^2 + a_1a_2^2a_5 - a_1a_2a_3a_4.$$

Page 397.

## Appendix 6. Optimum nodes of interpolation and number of Christoffel.

Standard nodes of Chebyshev and number of Christoffel for the uniform distribution of probabilities.

Table 1.

| $q$ | $k_j$ | $\lambda_{jk_j}$ | $\rho_{jk_j}$ |
|---|---|---|---|
| 1 | 1 | 0,0000000 | 1,0000000 |
| 2 | 1 | 0,5773503 | 0,5000000 |
| | 2 | —0,5773503 | 0,5000000 |
| 3 | 1 | 0,0000000 | 0,4444444 |
| | 2 | 0,7745967 | 0,2777778 |
| | 3 | —0,7745967 | 0,2777778 |
| 4 | 1 | 0,3399810 | 0,3260726 |
| | 2 | —0,3399810 | 0,3260726 |
| | 3 | 0,8611363 | 0,1759274 |
| | 4 | —0,8611363 | 0,1759274 |
| 5 | 1 | 0,0000000 | 0,2844444 |
| | 2 | 0,5384693 | 0,2393143 |
| | 3 | —0,5384693 | 0,2393143 |
| | 4 | 0,9061798 | 0,1184634 |
| | 5 | —0,9061798 | 0,1184634 |
| 6 | 1 | 0,2386192 | 0,2339570 |
| | 2 | —0,2386192 | 0,2339570 |
| | 3 | 0,6612094 | 0,1803808 |
| | 4 | —0,6612094 | 0,1803808 |
| | 5 | 0,9324695 | 0,0856622 |
| | 6 | —0,9324695 | 0,0856622 |
| 7 | 1 | 0,0000000 | 0,2089796 |
| | 2 | 0,4058452 | 0,1909150 |
| | 3 | —0,4058452 | 0,1909150 |
| | 4 | 0,7415312 | 0,1398527 |
| | 5 | —0,7415312 | 0,1398527 |
| | 6 | 0,9491079 | 0,0647425 |
| | 7 | —0,9491079 | 0,0647425 |
| 8 | 1 | 0,1834346 | 0,1813419 |
| | 2 | —0,1834346 | 0,1813419 |
| | 3 | 0,5255324 | 0,1568533 |
| | 4 | —0,5255324 | 0,1568533 |
| | 5 | 0,7966665 | 0,1111905 |
| | 6 | —0,7966665 | 0,1111905 |
| | 7 | 0,9602898 | 0,0506142 |
| | 8 | —0,9602898 | 0,0506142 |

Page 398.

Continuation Table 1.

| $q$ | $k_j$ | $\lambda_{jk_j}$ | $\rho_{jk_j}$ |
|---|---|---|---|
| 9 | 1 | 0,0000000 | 0,1651197 |
| | 2 | 0,3242534 | 0,1561735 |
| | 3 | —0,3242534 | 0,1561735 |
| | 4 | 0,6133714 | 0,1303053 |
| | 5 | —0,6133714 | 0,1303053 |
| | 6 | 0,83603[illegible] | 0,0903240 |
| | 7 | —0,8360311 | 0,0903240 |
| | 8 | 0,9681602 | 0,0406372 |
| | 9 | —0,9681602 | 0,0406372 |
| 10 | 1 | 0,1488743 | 0,1477621 |
| | 2 | —0,1488743 | 0,1477621 |
| | 3 | 0,4333954 | 0,1346334 |
| | 4 | —0,4333954 | 0,1346334 |
| | 5 | 0,6794096 | 0,1095432 |
| | 6 | —0,6794096 | 0,1095432 |
| | 7 | 0,8650634 | 0,0747257 |
| | 8 | —0,8650634 | 0,0747257 |
| | 9 | 0,9739065 | 0,0333357 |
| | 10 | —0,9739065 | 0,0333357 |

Standard nodes of Chebyshev and number of Christoffel for the normal law of distribution of probabilities.

Page 399. Table 2.

| $q$ | $k_j$ | $\lambda_{k_j}$ | $\rho_{k_j}$ |
|---|---|---|---|
| 1 | 1 | 0,0000000 | 1,0000000 |
| 2 | 1 | 0,9999999 | 0,5000000 |
| | 2 | —0,9999999 | 0,5000000 |
| 3 | 1 | 0,0000000 | 0,6666667 |
| | 2 | 1,7320508 | 0,1666667 |
| | 3 | —1,7320508 | 0,1666667 |
| 4 | 1 | 0,7419638 | 0,4541241 |
| | 2 | —0,7419638 | 0,4541241 |
| | 3 | 2,3344142 | 0,0458758 |
| | 4 | —2,3344142 | 0,0458758 |
| 5 | 1 | 0,0000000 | 0,5333333 |
| | 2 | 1,3556261 | 0,2220759 |
| | 3 | —1,3556261 | 0,2220759 |
| | 4 | 2,8569693 | 0,0112574 |
| | 5 | —2,8569693 | 0,0112574 |
| 6 | 1 | 0,6167066 | 0,4088284 |
| | 2 | —0,6167066 | 0,4088284 |
| | 3 | 1,8891759 | 0,08615746 |
| | 4 | —1,8891759 | 0,08615746 |
| | 5 | 3,3242574 | 0,002555784 |
| | 6 | —3,3242574 | 0,002555784 |
| 7 | 1 | 0,0000000 | 0,4571428 |
| | 2 | 1,1544053 | 0,2401231 |
| | 3 | —1,1544053 | 0,2401231 |
| | 4 | 2,3667594 | 0,03075712 |
| | 5 | —2,3667594 | 0,03075712 |
| | 6 | 3,7504397 | $0,5482688 \cdot 10^{-3}$ |
| | 7 | —3,7504397 | $0,5482688 \cdot 10^{-3}$ |
| 8 | 1 | 0,5390798 | 0,3730122 |
| | 2 | —0,5390798 | 0,3730122 |
| | 3 | 1,6365190 | 0,1172399 |
| | 4 | —1,6365190 | 0,1172399 |
| | 5 | 2,8024859 | $0,9635220 \cdot 10^{-2}$ |
| | 6 | —2,8024859 | $0,9635220 \cdot 10^{-2}$ |
| | 7 | 4,1445472 | $0,1126145 \cdot 10^{-3}$ |
| | 8 | —4,1445472 | $0,1126145 \cdot 10^{-3}$ |
| 9 | 1 | 0,0000000 | 0,4063492 |
| | 2 | 1,0232557 | 0,2440975 |
| | 3 | —1,0232557 | 0,2440975 |
| | 4 | 2,0768480 | 0,04901640 |
| | 5 | —2,0768480 | 0,04091640 |
| | 6 | 3,2054290 | $0,789141 \cdot 10^{-2}$ |
| | 7 | —3,2054290 | $0,789141 \cdot 10^{-2}$ |
| | 8 | 4,5127458 | $0,02234584 \cdot 10^{-3}$ |
| | 9 | —4,5127458 | $0,02234584 \cdot 10^{-3}$ |
| 10 | 1 | 0,4849357 | 0,3446423 |
| | 2 | —0,4849357 | 0,3446423 |
| | 3 | 1,4659891 | 0,1354837 |
| | 4 | —1,4659891 | 0,1354837 |
| | 5 | 2,4843258 | 0,01911158 |
| | 6 | —2,4843258 | 0,01911158 |
| | 7 | 3,5818235 | $0,758071 \cdot 10^{-3}$ |
| | 8 | —3,5818235 | $0,758071 \cdot 10^{-3}$ |
| | 9 | 4,8594628 | $0,4310653 \cdot 10^{-5}$ |
| | 10 | —4,8594628 | $0,4310653 \cdot 10^{-5}$ |

Pages 400-405.

## REFERENCES


1. Авербух А. И. О выборе расчетных случаев при анализе систем, содержащих случайные параметры. «Автоматика и телемеханика», т. XXIII, 1962, № 4.

2. Алгоритмический язык Алгол-60. Пересмотренное сообщение. Пер. с англ. под ред. А. П. Ершова, С. С. Лаврова и М. Р. Шура-Бура., М., «Мир», 1965.

3. Александров В. М., Батков А. М., Староверов А. М., Щукин Б. А. Исследование точности нелинейных нестационарных систем при помощи метода статистической линеаризации. «Автоматика и телемеханика», т. XXVI, 1965, № 3.

4. Андерсон Т. Введение в многомерный статистический анализ. М., Физматгиз, 1963.

5. Андреев Н. И. Оптимальные самонастраивающиеся системы. Труды III Международного конгресса по автоматическому управлению. М., Изд. АН СССР, 1968.

6. Андреев Н. И. Корреляционная теория статистически оптимальных систем. М., «Наука», 1966.

7. Андреев Н. И. Системы, самонастраивающиеся на минимум средней квадратической ошибки. «Известия АН СССР. Техническая кибернетика», 1966.

8. Андриенко А. Я. Метод задания возмущений при исследовании статистической точности нелинейных систем управления. В сб. «Техническая кибернетика», М., «Наука», 1965.

9. Бахвалов Н. С. К оценке ошибки при численном интегрировании дифференциальных уравнений экстраполяционным методом Адамса. ДАН СССР, т. 104, 1955, № 5.

10. Беллман Р. Динамическое программирование. М., Изд-во иностр. лит., 1960.

11. Березин И. С., Жидков Н. П. Методы вычислений, т. II, изд. 2-е, М., Физматгиз, 1962.

12. Боттенбрух Г. Структура АЛГОЛ-60 и его использование. М., Изд-во иностр. лит., 1963.

13. Бродский М. Л. Асимптотические оценки погрешностей при численном интегрировании дифференциальных уравнений разностными методами. ДАН СССР, т. 93, 1933, № 4.

14. Бунимович В. И. Флюктуационные процессы в радиоприемных устройствах. М., «Советское радио», 1951.

15. Бусленко Н. П., Голенко Д. И., Соболь И. М., Срагович В. Г. Шрейдер Ю. А. Метод статистических испытаний (метод Монте-Карло). М., Физматгиз, 1962.

16. Бусленко Н. П., Шрейдер Ю. А. Метод статистических испытаний (Монте-Карло) и его реализация в цифровых машинах. М., Физматгиз, 1961.

17. Бэкус Дж. В., Бауэр Ф. Л. и др. Сообщение об алгоритмическом языке АЛГОЛ-60 под ред. П. Наура. Пер. с англ. под ред. А. П. Ершова. М., «Журнал вычислительной математики и математической физики», т. 1, 1961, № 2.

18. Вентцель Е. С. Теория вероятностей, изд. 2-е. М., Физматгиз, 1962.
19. Верзунов М. В., Лобанов И. В., Семенов А. М. Однополосная модуляция. М., Связьиздат, 1962.
20. Виленкин Н. Я. О приближенном вычислении кратных интегралов. В сб. «Вычислительная математика», Сб. 5, М., Изд. АН СССР, 1959.
21. Гельфанд И. М., Вул Е. Б., Гинзбург С. Л., Федоров Ю. Г. Метод оврагов в задачах рентгеноструктурного анализа. М., «Наука», 1966.
22. Голдман С. Теория информации. М., Изд-во иностр. лит., 1957.
23. Голенко Д. И. Моделирование и статистический анализ псевдослучайных чисел на электронных вычислительных машинах. М., «Наука», 1965.
24. Databo Добродеев А. Н. Развитие метода аппроксимирующих полиномов для вычисления моментов выходных координат нелинейных систем. «Автоматика и телемеханика», т. XXVII, 1966, № 7.
25. Доступов Б. Г. Приближенное определение вероятностных характеристик выходных координат нелинейных систем автоматического регулирования. «Автоматика и телемеханика», т. XVIII, 1957, № 11.
26. Доступов Б. Г. Приближенное определение вероятностных характеристик решений системы обыкновенных дифференциальных уравнений, содержащих случайные параметры. Труды ВВИА им. Н. Е. Жуковского, вып. 592, 1956.
27. Доступов Б. Г., Пугачев В. С. Уравнение, определяющее закон распределения интеграла системы обыкновенных дифференциальных уравнений, содержащих случайные параметры. «Автоматика и телемеханика», т. XVIII, 1957, № 7.
28. Дубовицкий А. Я., Милютин А. А. Задачи на экстремум при наличии ограничений. «Журнал вычислительной математики и математической физики», т. 6, 1966, № 5.
29. Ефимова М. Н. Алгоритмические языки (обзор зарубежных работ). под ред. А. И. Китова. М., «Советское радио», 1965.
30. Зюко А. Г. Помехоустойчивость и эффективность систем связи. М., Связьиздат, 1963.
31. Казаков И. Е. Статистическая линеаризация однозначных и неоднозначных нелинейностей со случайными параметрами. Труды 3-го Всесоюзного совещания по автоматическому управлению. Статистические методы. М., «Наука», 1967.
32. Казаков И. Е. Обобщение метода статистической линеаризации на многомерные системы. «Автоматика и телемеханика», т. XXVI, 1965, № 7.
33. Казаков И. Е. Приближенный метод статистического исследования нелинейных систем. Труды ВВИА им. Н. Е. Жуковского, вып. 394, 1954.
34. Казаков И. Е., Доступов Б. Г. Статистическая динамика нелинейных автоматических систем. М., Физматгиз, 1962.
35. Казаков И. Е. Словцов Н. И. Применение метода моментов к исследованию точности динамических систем при случайных возмущениях в виде многочленов со случайными параметрами. «Автоматика и телемеханика», т. XXIX, № 11, 1968.
36. Канторович Л. В. Функциональный анализ и прикладная математика. «Успехи математических наук», III вып., 1948.
37. Келли Г. Дж. Метод градиентов. В сб. «Методы оптимизации с приложениями к механике космического полета». «Наука», 1965.
38. Китов А. И. Программирование информационно-логических задач. М., «Советское радио», 1967.
39. Козлов М. С., Федоренко Г. И. Динамика самонастраивающейся системы управления полетом, сохраняющей заданный запас устойчивости. В сб. «Самонастраивающиеся системы». М., «Наука», 1965.

40. Коллатц Л. К. Численные методы решения дифференциальных уравнений. Пер. с нем. М., Изд-во иностр. лит., 1953.
41. Корн Г. А. Моделирование случайных процессов на аналоговых и аналого-цифровых машинах. Пер. с англ. М., «Мир», 1968.
42. Королев М. А., Кузьмин К. С., Лавров С. С., Летичевский А. А. Сообщение об алгоритмическом языке АЛГЭК. «Кибернетика», АН УССР, 1966, № 2.
43. Котельников В. А. Теория потенциальной помехоустойчивости. М., Госэнергоиздат, 1956.
44. Коршунов Ю. М. О построении эквивалентного комплексного коэффициента нелинейного импульсного элемента. «Автоматика и телемеханика», т. XXIII, 1962, № 5.
45. Кочетков В. Т., Пономарев В. М., Половко В. М. Теория систем телеуправления и самонаведения ракет. М., «Наука», 1964.
46. Кочетков В. Т., Поцелуев А. В. Статистический синтез дискретных нелинейных систем управления. «Известия АН СССР, Техническая кибернетика», 1968, № 4.
47. Кочетков В. Т., Поцелуев А. В. О синтезе оптимальной нелинейной системы управления по критерию «максимума вероятности». «Известия АН СССР, Техническая кибернетика», 1966, № 4.
48. Крамер Г. Математические методы статистики. М., Изд-во иностр. лит., 1948.
49. Красовский А. А. Динамика непрерывных самонастраивающихся систем. М., Физматгиз, 1963.
50. Крутько П. Д. Статистическая динамика импульсных систем. М., «Советское радио», 1963.
51. Крутько П. Д. Задача определения дискретного формирующего фильтра. «Автоматика и телемеханика», т. XXII, 1961, № 11.
52. Крылов В. И., Шульгина Л. Т. Справочная книга по численному интегрированию. М., «Наука», 1966.
53. Кузин Л. Т. Расчет и проектирование дискретных систем управления. М., Машгиз, 1962.
54. Кузнецов П. И., Стратонович Р. Л., Тихонов В. И. Прохождение случайных функций через нелинейные системы. «Автоматика и телемеханика», т. XIV, 1953, № 4.
55. Кульман Н. К. Оптимальное выделение амплитудно-модулированного сигнала из шумов при помощи синхронного детектирования. «Радиотехника и электроника», 1964, № 5.
56. Кульман Н. К., Стратонович Р. Л. Фазовая автоподстройка частоты и оптимальное намерение параметров узкополосного сигнала с непостоянной частотой в шуме. «Радиотехника и электроника», 1964, № 1.
57. Лавров С. С. Универсальный язык программирования (АЛГОЛ-60). М., «Наука», 1964.
58. Ланс Дж. Н. Численные методы для быстродействующих вычислительных машин. М., Изд-во иностр. лит., 1962.
59. Левитин Е. С., Поляк Б. Т. Методы минимизации при наличии ограничений. «Журнал вычислительной математики и математической физики», т. 6, 1966, № 5.
60. Лэнинг Дж. Х., Бэттин Р. Г. Случайные процессы в задачах автоматического управления. Пер. с англ. М., Изд-во иностр. лит., 1958.
61. Любимов Ю. К. Получение на ЦВМ дискретных значений стационарного случайного процесса в неравноотстоящих точках. «Автоматика и телемеханика», XXVI, 1965, № 12.
62. Милн В. Э. Численное решение дифференциальных уравнений. Пер. с англ. под ред. М. Р. Шура-Бура. М., Изд-во иностр. лит., 1955.
63. Михлин С. Г. Численная реализация вариационных методов. М., «Наука», 1966.

64. Мишкин Э., Браун Л. Приспосабливающиеся автоматические системы. М., Изд-во иностр. лит., 1963.

65. Налимов В. В., Чернова Н. А. Статистические методы планирования экстремальных экспериментов. М., «Наука», 1965.

66. Немировский М. С. Помехоустойчивость радиосвязи. М., «Энергия», 1966.

67. Никольс М. Х., Раух Л. А. Радиотелеметрия. М., Изд-во иностр. лит., 1958.

68. Оплер. Новые направления в разработке средств математического обеспечения 1960—1966 гг. Труды института инженеров по электротехнике и радиоэлектронике «Вычислительные машины», т. 54, 1966, № 12. Русский пер. М., «Мир», 1967.

69. Первозванский А. А. Случайные процессы в нелинейных автоматических системах. М., Физматгиз, 1962.

70. Петров И. И. К вопросу сравнительной оценки систем связи с различными видами модуляции. «Радиотехника», 1967, № 4.

71. Питерсон У. Коды, исправляющие ошибки. М., «Мир», 1964.

72. Поляк Б. Т. Градиентные методы минимизации функционалов. «Журнал вычислительной математики и математической физики», т. 3, 1963, № 4.

73. Пономарев В. М. Приближенный синтез оптимального управления, Труды III Всесоюзного совещания по автоматическому управлению (Технической кибернетике), М., «Наука», 1967.

74. Понтрягин Л. С., Болтянский В. Г., Гамкрелидзе Р. В., Мищенко Е. Ф. Математическая теория оптимальных процессов. М., Физматгиз, 1961.

75. Попов Е. П., Пальтов И. П. Приближенные методы исследования нелинейных автоматических систем. М., Физматгиз, 1960.

76. Пугачев В. С. и др. Основы автоматического управления. М., Физматгиз, 1963.

77. Пугачев В. С. Теория случайных функций и ее применение к задачам автоматического управления. Изд. 3-е. М., Физматгиз, 1962.

78. Пугачев В. С. О приближенном представлении случайных функций в виде линейных комбинаций случайных величин. В сб. научных трудов ВВИА им. Н. Е. Жуковского, т. II, 1957.

79. Райс Д. О. Теория флюктуационных шумов. Сб. переводов под ред. Н. А. Железнова. М., Изд-во иностр. лит., 1953.

80. Ракитский Ю. В. О некоторых свойствах решений систем обыкновенных дифференциальных уравнений одношаговыми методами численного интегрирования. «Журнал вычислительной математики и математической физики», т. 1, 1961, № 6.

81. Раков Г. К. Выработка случайной коррелированной величины на быстродействующей электронной счетной машине, изд. ИТМ и ВТ АН СССР, М., 1959.

82. Растригин Л. А. Случайный поиск в задачах оптимизации. Рига, «Зинатне», 1965.

83. Сандерс. Сравнение эффективности некоторых систем связи. «Зарубежная радиоэлектроника», 1960, № 12.

84. Сейдж, Смит. Управление системами с помощью цифрового моделирования в реальном масштабе времени. Труды института инженеров по электротехнике и радиоэлектронике. «Вычислительные машины», т. 54, 1966, № 12, Русский пер. М., «Мир», 1967.

85. Смирнов Н. В., Дунин-Барковский И. В. Краткий курс математической статистики. М., Физматгиз, 1959.

86. Современная математика для инженеров. Пер. с англ. под ред. Э. Ф. Беккенбаха. М., Изд-во иностр. лит., 1958.

87. Современное программирование. Сборник статей. Пер. с англ., М., «Советское радио», вып. 1, 1966, вып. 2, 1967.
88. Солодов А. В. Линейные системы автоматического управления с переменными параметрами. М., Физматгиз, 1962.
89. Солодовников В. В. Статистическая динамика линейных систем автоматического управления. М., Физматгиз, 1960.
90. Сраговнч В. Г. Моделирование некоторых классов случайных процессов. «Журнал вычислительной математики и математической физики», т. 3, 1963, № 3.
91. Стратонович Р. Л. Избранные вопросы теории флюктуаций в радиотехнике. М., «Сов. радио», 1961.
92. Стратонович Р. Л. Условные процессы Маркова. «Теория вероятностей и ее применения», 1960, вып. 2.
93. Стратонович Р. Л. Применение теории процессов Маркова для оптимальной фильтрации сигналов. «Радиотехника и электроника», 1960, вып. 11.
94. Теория ортогональных рядов. Пер. с нем. М., Физматгиз, 1958.
95. Теterич Н. М. Генераторы шума. М.— Л., Госэнергоиздат, 1961.
96. Тихонов В. И. Статистическая радиотехника. М., «Советское радио», 1966.
97. Тиц А. В. Приближенное вычисление $n$-кратных интегралов. «Известия АН СССР, серия математическая», 1940, № 4.
98. Трахтенгерц Э. А. Языки программирования. «Автоматика и телемеханика», т. XXVI, 1965, № 6.
99. Уорсинг А., Геффнер Дж. Методы обработки экспериментальных данных. М., Изд-во иностр. лит., 1953.
100. Фельдбаум А. А. Основы теории оптимальных автоматических систем. М., «Наука», 1966.
101. Харкевич А. А. Борьба с помехами. М., Физматгиз, 1965.
102. Цикин И. А. О помехоустойчивости систем связь ОБП и ДБП. «Радиотехника», 1962, № 4.
103. Цирлер Н. Линейные возвратные последовательности. «Кибернетический сборник», вып. 6. М., Изд-во иностр. лит., 1963.
104. Цыпкин Я. З. Теория линейных импульсных систем. М., Физматгиз, 1963.
105. Чернецкий В. И. Анализ точности нелинейных систем управления, М., «Машиностроение», 1968.
106. Чернецкий В. И. Об одном методе исследования качества нелинейных стохастических систем автоматического управления. Сб. «Автоматическое управление» под ред. Б. Н. Петрова. «Наука», 1968.
107. Чернецкий В. И. Методы вычисления вероятностных характеристик нелинейных систем автоматического управления. В сб. «Автоматическое управление и вычислительная техника», вып. 5, М., Машгиз, 1962.
108. Чернецкий В. И. Определение функций распределения вероятностей на выходе нелинейных систем управления. Изд. АН СССР. ОТН «Энергетика и автоматика», 1961, № 2.
109. Чернецкий В. И. Интерполяционный метод анализа точности систем автоматического регулирования при случайных воздействиях. «Автоматика и телемеханика», 1960, т. XXI, № 4.
110. Чинаев П. И. Самонастраивающиеся системы. М., Машгиз, 1963.
111. Шахмундес Ю. Л. К вопросу о близости движений исследуемого объекта к его цифровой модели. В сб. «Вычислительная техника в управлении». М., «Наука», 1966.
112. Шеннон К. Работы по теории информации и кибернетике. М., Изд-во иностр. лит., 1963.
113. Шура-Бура М. Р. Оценка ошибок численного интегрирования обыкновенных дифференциальных уравнений. «Прикладная математика и механика», 1952, т. 16, № 5.

114. Эшби У. Схема усилителя мыслительных способностей. В сб «Автоматы», под ред. К. Э. Шеннона и Дж. Маккарти. Пер. с англ., М., изд-во иностр. лит., 1956.
115. Юдин Д. Б., Гольдштейн Е. Г. Линейное программирование, М., Физматгиз, 1963.
116. Юдовский О. В. Датчики случайных чисел с автоматической коррекцией. Труды Всесоюзного нефтегаз НИИ, 1966, вып. 47.
117. Box G. E. P., Wilson K. B. On the Experimental Attainement of Optimum Conditions, Journal of the Royal Statistical Society, Series B, 1951, vol. 13, № 1.
118. Chin T. H. Spacecraff stabilisation and attitude control. Space Aeronautics 1963, V39, № 6, p. 88—93.
119. Costas J. P. Synchronous communications. «Proc. IRE», 1956, № 12
120. Dantzig G. B. The Dual Simplex Algorithm. RAND Report RM-1270 The RAND Corporation, Santa Monica, Calif., 1954.
121. Dantzig G. B. Linear Programming under Uncertainty. RAND Report RM-1374. The RAND Corporation, Santa Monica, Calif. 1954.
122. Dunkan D. B. Response of linear time-dependent Systems to random inputs. J. Appl. Phys., 24, May 1963.
123. Groos O., Johnson S. M. Additive generation of pseudo-random numbers, Rand Corporation, 1958.
124. Kuo F. F., Freeny S. L. Hilbert transforms and modulation theory Proc. NEC, 1962, 18.
125. Lehmer D. H. Mathematical methods in lagre scale computing units. Annals Comp. Laboratory Harvard Univ., 26, 1951.
126. Lindsey W. C. Optimal cohrent linear demodulation «IEEE Trans on communication Technology», 1965, № 2.
127. Neumann J. Various Techniques used in connection with random digits. NBS Appl. Math. Series 12, 1951.
128. Painter J. H. A combined single sideband hybrid AM-PM signal «Proc. IEEE», 1966, № 7.
129. Papoulis A. Probility. random variables and stochastic processes. *McGraw-Hill, 1965.*
130. Rotenberg A. A new pseudo-random number generator J. Assoc. Comp. Mach., 7 (1960).
131. Scott E. D. Pseudorate Sawtooth Pulse-Reset Control System Analysis and Design, Journal of Spacecraft and Rocets, v. 4, № 6, June, 1967.
132. Thomas J., Williams T., Wolf J. The demodulation of AM—signals in noise. Proc. of 3-d National Convention on Military Electronics, 1959.
133. Viterby A. J. Optimum coherent demodulation for continuous modulation systems. Proc. NEC, 1962, 18.
134. Youla D. C. The use of the method of maximum likelihood in estimating continuons-modulated intelligence which has been corrupted by noise. «IRE Trans. on Information theory», 1964, PGIT-3.

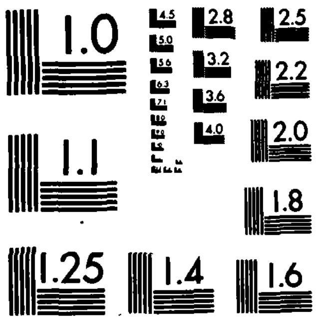
Page 406.

The subject index.

AD-A143 297 STATIC METHODS IN THE DESIGN OF NONLINEAR AUTOMATIC CONTROL SYSTEMS(U) FOREIGN TECHNOLOGY DIV WRIGHT-PATTERSON AFB OH N I ANDREYEV ET AL. 27 JUN 84 9/9

UNCLASSIFIED FTD-ID(RS)T-1734-83 F/G 12/1 NL

END




MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

END

FILMED

9-84

DTI